

PROXIMITY TO ℓ_p AND c_0 IN BANACH SPACES

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ABSTRACT. We construct a class of minimal trees and use these trees to establish a number of coloring theorems on general trees. Among the applications of these trees and coloring theorems are quantification of the Bourgain ℓ_p and c_0 indices, dualization of the Bourgain c_0 index, establishing sharp positive and negative results for constant reduction, and estimating the Bourgain ℓ_p index of an arbitrary Banach space X in terms of a subspace Y and the quotient X/Y .

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1. INTRODUCTION

Withint the class of Banach spaces, the sequence spaces ℓ_p and c_0 play a central role. James [18] famously showed that every Banach space which contains isomorphic copies of either ℓ_1 or c_0 must contain almost isometric copies of these spaces. With Odell and Schlumprecht's solution of the distortion problem [24], it was shown that the corresponding result for ℓ_p , $1 < p < \infty$, is false. However, it follows from Krivine's theorem [20] that if ℓ_p , $1 < p < \infty$, is crudely finitely representable in X , then ℓ_p is finitely representable in X . The corresponding result for ℓ_1 and c_0 was also shown by James in [18]. Thus we see a difference between the weakest form of admitting subspaces of a Banach space admitting ℓ_p structure, finite representability of ℓ_p in X , and the strongest notion, admitting isomorphic copies of ℓ_p as a subspace. The Bourgain ℓ_p index [7] allows a quantification which bridges the gap

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between these two forms of admitting ℓ_p structures within a given Banach space. In [19], transfinite versions of the argument of James were given for countable ordinals which allow a unified approach to the two previously mentioned results of James. There, they also gave an argument that the transfinite versions of the argument must fail for the ℓ_p index when $1 < p < \infty$ must break down somewhere between the positive results from Krivine, which represent the case of an index at least ω , the smallest infinite ordinal, and the negative results which follow from Odell and Schlumprecht, and they asked for a quantification of this result.

Another well-known fact is that if X, Y are Banach spaces admitting no subspace isomorphic to ℓ_p , then $X \oplus Y$ admits no subspace isomorphic to ℓ_p . Dodos [12] prove that, in the sense of the Bourgain ℓ_p index, this result is uniform. That is, there exists a function $\phi_p : [1, \omega_1)^2 \rightarrow [1, \omega_1)$ so that if X, Y have ℓ_p indices ξ, ζ , respectively, then the Bourgain ℓ_p index of $X \oplus Y$ cannot exceed $\phi_p(\xi, \zeta)$. Dodos asked for an explicit estimate on the function ϕ_p .

This paper is arranged as follows. In Section 2, we discuss trees, derivations, and orders and establish several standard methods that we will use throughout to witness estimates of the Bourgain ℓ_p index of a Banach space. In Section 3, we establish a number of coloring lemmas of independent interest which yield certain dichotomies on trees. In Section 4, we define the different types of ℓ_p and c_0 structures we wish to quantify and the classes of Banach spaces which will be our primary objects of study. In Section 5, we discuss the dualization of the Bourgain ℓ_p indices. In Sections 6 and 7, we discuss positive and negative results concerning the transfinite analogues of the equivalence of containment of crude ℓ_p and almost isometric ℓ_p structures mentioned above. More precisely, in Section 6 we given precise positive results for ℓ_1 and c_0 structures and establish the sharpness of these results. In Section 7, we discuss the implications of Krivine's theorem in producing positive results for $1 < p < \infty$, and quantify the Odell-Schlumprecht distortion of ℓ_p to answer the question raise in [19] of quantifying where the positive results may fail. Finally, in Section 8, we provide explicit estimates on the function ϕ_p of Dodos. More generally, we provide estimates of the Bourgain ℓ_p index of an arbitrary Banach space in terms of the Bourgain ℓ_p indices of a subspace and the corresponding quotient. We use this result to estimate the Bourgain ℓ_p index of infinite direct sums of Banach spaces.

2. TREES, DERIVATIONS, AND ORDER

Given a set S , we let $S^{<\omega}$ denote the finite sequences in S . We include in $S^{<\omega}$ the empty sequence, denoted \emptyset . Given $s \in S^{<\omega}$, we let $|s|$ denote the length of s . If $s = (x_i)_{i=1}^k$ and $0 \leq n \leq k$, we let $s|_n = (x_i)_{i=1}^n$. If $s, t \in S^{<\omega}$, we let $s \hat{\ } t$ denote the concatenation of s with t . We order $S^{<\omega}$ with the partial order \prec by letting $s \prec t$ if and only if s is a proper initial segment of t . In this case, we say s is a *predecessor* of t and that t is a *successor* or *extension* of s .

We say a subset $T \subset S^{<\omega}$ is a *tree on S* (or simply a *tree*) provided it is downward closed with respect to the order \prec . We say a tree is *hereditary* provided it contains all subsequences of its members. If S is a topological space, we say T is a *closed tree* if for each $n \in \mathbb{N}$, $T \cap S^n$ is a closed set. Here, S^n is endowed with the product topology. If A is a subset of a vector space, and if $1 \leq p \leq \infty$, we say x is a *p -absolutely convex combination* of A if there exists $(x_i)_{i=1}^n \in A^{<\omega}$ and $(a_i)_{i=1}^n \in S_{\ell_p^n}$ so that $x = \sum_{i=1}^n a_i x_i$. The *p -absolutely convex hull* of A is the set of all p -absolutely convex combinations of A and is denoted by $\text{co}_p(A)$. We say $(y_i)_{i=1}^m$ is a *p -absolutely convex block* of $(x_i)_{i=1}^n$ if there exist $0 = k_0 < \dots < k_m = n$ so that for each $1 \leq i \leq m$, $y_i \in \text{co}_p(x_j : k_{i-1} < j \leq k_i)$. We say the tree T is *p -absolutely convex* provided it contains all p -absolutely convex blocks of its members.

If $T \subset S^{<\omega}$ and if $s \in S^{<\omega}$, we let $T(s) = \{t \in S^{<\omega} : s \hat{~} t \in T\}$. If T is a tree, $T(s)$ is as well. If $U \subset S^{<\omega}$, we let $C(U)$ denote all finite, non-empty subsets of $U \setminus \{\emptyset\}$ which are linearly ordered with respect to the order \prec , and we call the members of $C(U)$ the *segments* of U . We order $C(U)$ by letting $c < c'$ if $s \prec t$ for all $s \in c$ and $t \in c'$. We also order $C(U) \cup \{\emptyset\}$ by declaring $\emptyset < c$ for each $c \in C(U)$. If P, Q are partially ordered sets with orders $<_P, <_Q$, respectively, we say a function $f : P \rightarrow Q$ is *order preserving* if for each $x, y \in P$ with $x <_P y$, $f(x) <_Q f(y)$. We emphasize that this is not an if and only if. That is, if x, y are incomparable in P , $f(x), f(y)$ need not be incomparable in Q in order for f to be order preserving. Of course, the composition of order preserving maps is order preserving. If $i : \mathcal{T} \rightarrow \mathcal{W}$ is order preserving, then the image of a segment under i is also a segment. Thus for any $U \subset \mathcal{T}$, i induces a map, which we also call i , between $C(U)$ and $C(i(U))$. That is, for $c \in C(U)$, $i(c) = \{i(t) : t \in c\}$. We say $f : P \rightarrow Q$ is an *embedding* if for each $x, y \in P$, $x <_P y$ if and only if $f(x) <_Q f(y)$.

We let $\text{MAX}(T)$ denote the maximal members of T with respect to the order \prec . We let $T' = T \setminus \text{MAX}(T)$, and note that if T is a tree, T' is a tree as well. For $\xi \in \mathbf{Ord}$, the class of ordinal numbers, we define the transfinite derived trees $d^\xi(T)$ of T by

$$d^0(T) = T,$$

$$d^{\xi+1}(T) = (d^\xi(T))',$$

and if ξ is a limit ordinal,

$$d^\xi(T) = \bigcap_{\zeta < \xi} d^\zeta(T).$$

Note that for any set S and any tree T on S , there must exist $\xi \in \mathbf{Ord}$ so that $d^{\xi+1}(T) = d^\xi(T)$. We say that T is *well-founded* if $d^\xi(T) = \emptyset$ for some ξ , and *ill-founded* otherwise. We let

$$o(T) = \min\{\xi \in \mathbf{Ord} : d^\xi(T) = \emptyset\},$$

where, as a matter of convenience, we write $\min \emptyset = \infty$. That is, $o(T) = \infty$ means that T is ill-founded. Also as a matter of convenience, we declare that $\xi < \infty$ for any $\xi \in \mathbf{Ord}$, and that $\xi\infty = \infty = \infty\xi$, and $\infty + \xi = \infty = \xi + \infty$. Note that if $o(T) < \infty$, $o(T)$ must be a

successor, since $\emptyset \in d^\xi(T)$ whenever $d^\xi(T)$ is non-empty. We observe that T is ill-founded if and only if there exists an infinite sequence $(x_i) \subset S$ so that for all $n \in \mathbb{N}$, $(x_i)_{i=1}^n \in T$.

We say $T \subset S^{<\omega} \setminus \{\emptyset\}$ is a B -tree provided $T \cup \{\emptyset\}$ is a tree. For a number of our purposes, it will be convenient to avoid having to include the empty sequence in our considerations. Note that all notions above concerning trees have obvious analogous definitions for B -trees.

The following items are either trivial or easily shown by induction.

Proposition 2.1. *Let S be a set and let T be a tree on S .*

- (i) *For any $\xi \in \mathbf{Ord}$ and $s \in S^{<\omega}$, $d^\xi(T)(s) = d^\xi(T(s))$.*
- (ii) *For any $\xi, \zeta \in \mathbf{Ord}$, $d^\zeta(d^\xi(T)) = d^{\xi+\zeta}(T)$.*
- (iii) *For any $s \in S^{<\omega}$ and $\xi \in \mathbf{Ord}$, $\xi < o(T(s))$ if and only if $s \in d^\xi(T)$. In particular, if $T(s)$ is well-founded, $o(T(s)) = \max\{\xi \in \mathbf{Ord} : s \in d^\xi(T)\} + 1$.*

Moreover, (i) and (ii) hold if T is a B -tree, and (iii) does as well as long as $s \neq \emptyset$.

We will also want standard ways to construct new B -trees with prescribed orders from given B -trees of specified orders. The first method will be to construct a tree of order $\xi + \zeta$ by placing a “copy” of a tree of order ξ beneath every maximal member of a B -tree of order ζ . The second method will be to take a totally incomparable union of B -trees with orders having supremum ξ .

Proposition 2.2. (i) *Let A, B be disjoint sets and let $\xi, \zeta \in \mathbf{Ord}$. Suppose that T_\emptyset is a B -tree on A of order ζ and that for each $t \in \text{MAX}(T_\emptyset)$, T_t is a B -tree on B of order ξ . Then*

$$T = T_\emptyset \cup \{t \hat{\ } s : t \in \text{MAX}(T_\emptyset), s \in T_t\}$$

is a B -tree on $A \cup B$ of order $\xi + \zeta$.

- (ii) *Fix a limit ordinal ξ and a subset $A \subset [0, \xi)$ such that $\sup A = \xi$. Suppose that for each $\zeta \in A$, there exists $A_\zeta \subset [0, \xi)$ so that the sets $(A_\zeta)_{\zeta \in A}$ are pairwise disjoint. Suppose that for each $\zeta \in A$, T_ζ is a B -tree on A_ζ with $o(T_\zeta) = \zeta$. Then*

$$T = \bigcup_{\zeta \in A} T_\zeta$$

is a B -tree on $[0, \xi)$ with order ξ .

- (iii) *Fix a limit ordinal ξ and a subset $A \subset [0, \xi)$ such that $\sup A = \xi$. Suppose that for each $\zeta \in A$, $T_\zeta \subset [0, \zeta)$ is a B -tree of order ζ . Then*

$$T = \bigcup_{\zeta \in A} \{(\zeta), (\zeta) \hat{\ } t : t \in T_\zeta\}$$

is a B -tree on $[0, \xi)$ with order ξ .

Proof. (i) Note that for each $t \in \text{MAX}(T_\emptyset)$, $T(t) = T_t \cup \{\emptyset\}$. Therefore using the correspondence $s \leftrightarrow t \hat{\ } s$, we deduce $d^\xi(T(t)) = \{\emptyset\}$, so that $t \in \text{MAX}(d^\xi(T))$. Thus $d^\xi(T) = T_\emptyset$, whence $o(T) = \xi + \zeta$.

(ii) Note that the derived set of the union is simply the union of the derived sets, so that for each $\eta < \xi$,

$$d^\eta(T) = \bigcup_{\zeta \in A} d^\eta(T_\zeta).$$

(iii) Note that the B -trees $S_\zeta = \{(\zeta), (\zeta)^\frown t : t \in T_\zeta\}$ are totally incomparable for distinct ζ . This means $d^\eta(T) = \bigcup_{\zeta \in A} d^\eta(S_\zeta)$ for each $\eta < \xi$. It is clear, or follows from (i), that $o(S_\zeta) = o(T_\zeta) + 1 = \zeta + 1 < \xi$. Thus for any $\eta < \xi$, there exists $\zeta \in A$ with $\eta < \zeta$, and so $d^\eta(T) \supset d^\eta(S_\zeta) \neq \emptyset$. But $d^\xi(T) = \bigcup_{\zeta \in A} d^\xi(S_\zeta) = \emptyset$.

□

For convenience, we will define for each $\xi \in \mathbf{Ord}$ a tree \mathcal{MT}_ξ on $[1, \xi]$ which will be minimal in a sense which is made clear in the proposition below. More precisely, we construct \mathcal{MT}_ξ by induction on ξ so that \mathcal{MT}_ξ is a tree on $[1, \xi]$ consisting of strictly decreasing sequences in $[1, \xi]$. These trees can be compared to the trees constructed in [19]. There a different definition of tree was used. The sets they define are partially ordered by set inclusion, rather than our order \prec .

We let

$$\mathcal{MT}_0 = \{\emptyset\},$$

$$\mathcal{MT}_{\xi+1} = \{\emptyset\} \cup \{(\xi+1)^\frown t : t \in \mathcal{MT}_\xi\},$$

and if $\xi \in \mathbf{Ord}$ is a limit ordinal,

$$\mathcal{MT}_\xi = \bigcup_{\zeta < \xi} \mathcal{MT}_{\zeta+1}.$$

For each $\xi \in \mathbf{Ord}$, we let $\mathcal{T}_\xi = \mathcal{MT}_\xi \setminus \{\emptyset\}$. Note that for a limit ordinal ξ , $\mathcal{T}_\xi = \bigcup_{\zeta < \xi} \mathcal{T}_{\zeta+1}$, and that the B -trees in this union are totally incomparable. This is because every member of $\mathcal{T}_{\zeta+1}$ is an extension of $(\zeta+1)$.

Proposition 2.3. *Let $\xi \in \mathbf{Ord}$. Let S be a set and let T be a B -tree on S .*

- (i) \mathcal{T}_ξ is a B -tree on $[1, \xi]$ with $o(\mathcal{T}_\xi) = \xi$.
- (ii) If there exists $f : \mathcal{T} \rightarrow T$ which is order preserving, $o(T) \geq o(\mathcal{T})$.
- (iii) $o(T) \geq \xi$ if and only if there exists an order preserving $f : \mathcal{T}_\xi \rightarrow T$.
- (iv) $o(T) \geq \xi$ if and only if there exists a function $f : \mathcal{T}_\xi \rightarrow S$ so that for each $t \in \mathcal{T}_\xi$, $(f(t|_i))_{i=1}^{|t|} \in T$.

Proof. (i) That \mathcal{T}_ξ is a B -tree on $[1, \xi]$ is trivial. That $o(\mathcal{T}_\xi) = \xi$ follows from Proposition 2.2.

(ii) We claim that for each $\zeta < o(\mathcal{T})$ and each $t \in d^\zeta(\mathcal{T})$, $f(t) \in d^\zeta(T)$. The proof is by induction on ζ with the base case and limit ordinal case trivial. If the claim holds for ζ and if $\zeta+1 < o(\mathcal{T})$, then for $t \in d^{\zeta+1}(\mathcal{T})$ we can choose $t \prec s \in d^\zeta(\mathcal{T})$. Then $f(t) \prec f(s) \in d^\zeta(T)$ and $f(t) \in d^{\zeta+1}(T)$.

We prove (iii) and (iv) together. First, if $f : \mathcal{T}_\xi \rightarrow S$ is such that for each $t \in \mathcal{T}_\xi$, $(f(t|_i))_{i=1}^{|t|} \in T$, then $g(t) = (f(t|_i))_{i=1}^{|t|}$ defines an order preserving map of \mathcal{T}_ξ into T , so that $o(T) \geq o(\mathcal{T}_\xi) = \xi$. It remains only to show that if $o(T) \geq \xi$, there exists such an f . If $\xi = 0$, there is nothing to prove. Suppose we have established the result for some ξ and that $o(T) \geq \xi + 1$. Then this means $d^\xi(T)$ cannot be empty, and therefore that there exists some $x \in S$ so that $(x) \in d^\xi(T)$. But by Proposition 2.1, $o(T(x)) \geq \xi$. This means there exists $f' : \mathcal{T}_\xi \rightarrow S$ so that for each $t \in \mathcal{T}_\xi$, $(f'(t|_i))_{i=1}^{|t|} \in T(x)$. Define $f : \mathcal{T}_{\xi+1} \rightarrow S$ by letting $f((\xi+1)) = x$ and $f((\xi+1)^\frown t) = f'(t)$ for $t \in \mathcal{T}_\xi$. This map is clearly seen to satisfy the requirements.

Last, suppose the result holds for every $\zeta < \xi$, ξ a limit ordinal, and suppose $o(T) \geq \xi$. Then $o(T) \geq \zeta + 1$ for every $\zeta < \xi$, so there exists $f_\zeta : \mathcal{T}_{\zeta+1} \rightarrow S$ so that for each $t \in \mathcal{T}_{\zeta+1}$, $(f_\zeta(t|_i))_{i=1}^{|t|} \in T$. Define f on \mathcal{T}_ξ by letting $f|_{\mathcal{T}_{\zeta+1}} = f_\zeta$.

□

2.1. Regular families. If M is any subset of \mathbb{N} , we let $[M]$ (resp. $[M]^{<\omega}$) denote the infinite (resp. finite) subsets of M . Throughout, we identify subsets of \mathbb{N} with strictly increasing sequences in \mathbb{N} in the obvious way. In this way, we can identify $[\mathbb{N}]^{<\omega}$ with a subset of $\mathbb{N}^{<\omega}$, and each of the definitions from the previous section concerning trees and B -trees can be applied to $[\mathbb{N}]^{<\omega}$ with this identification. In particular, $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is hereditary if it contains all subsets of its members. For $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$, we let $\widehat{\mathcal{F}} = \mathcal{F} \setminus \{\emptyset\}$.

We also identify $[\mathbb{N}]^{<\omega}$ with a subset of the Cantor set $2^\mathbb{N}$ by identifying sets with their indicator functions. We endow $[\mathbb{N}]^{<\omega}$ with the topology it inherits from this identification, and we say $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is compact if it is compact with respect to this topology.

If $(j_i)_{i=1}^l, (k_i)_{i=1}^l \in [\mathbb{N}]^{<\omega}$ are such that $j_i \leq k_i$ for each $1 \leq i \leq l$, then we say $(k_i)_{i=1}^l$ is a *spread* of $(j_i)_{i=1}^l$. We say $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is *spreading* provided it contains all spreads of its members. We say $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is *regular* provided it is compact, spreading, and hereditary.

We note that if \mathcal{F} is spreading, $MAX(\mathcal{F})$ is the set of isolated points in \mathcal{F} , and \mathcal{F}' is the Cantor-Bendixson derivative of \mathcal{F} . Note that \mathcal{F}' is also regular. For \mathcal{F} regular, we define

$$\iota(\mathcal{F}) = \min\{\xi < \omega_1 : d^\xi(\mathcal{F}) \subset \{\emptyset\}\}.$$

Note that since \mathcal{F} is countable compact Hausdorff, \mathcal{F} must have countable Cantor-Bendixson index. If $\emptyset \neq \mathcal{F}$, then $d^{\iota(\mathcal{F})}(\mathcal{F}) = \{\emptyset\}$ and $\iota(\mathcal{F}) + 1$ is the Cantor-Bendixson index of \mathcal{F} .

We write $E < F$ provided $\max E < \min F$. We write $n < E$ (resp. $n \leq E$) provided $n < \min E$ (resp. $n \leq \min E$). By convention, $\emptyset < E < \emptyset$ for all $E \in [\mathbb{N}]^{<\omega}$. If $E < F$, we write $E^\frown F$ in place of $E \cup F$. We say $(E_i)_{i=1}^n$ is \mathcal{F} -admissible provided $E_1 < \dots < E_n$ and $(\min E_i)_{i=1}^n \in \mathcal{F}$.

If $M \in [\mathbb{N}]$, there is a natural bijection between $2^\mathbb{N}$ and 2^M , which we also denote by M . That is, if $M = (m_n)_{n \in \mathbb{N}}$, $M(E) := (m_n : n \in E)$. We let $\mathcal{F}(M) = \{M(E) : E \in \mathcal{F}\}$.

If $(\mathcal{G}_i)_{i=1}^n$ are regular, then

$$(\mathcal{G}_1, \dots, \mathcal{G}_n) := \{E_1^\frown \dots^\frown E_n : E_i \in \mathcal{G}_i \ \forall 1 \leq i \leq n\}$$

is regular. For \mathcal{G} regular and $n \in \mathbb{N}$, we let $(\mathcal{G})_n = \mathcal{G}$ if $n = 1$ and $(\mathcal{G})_n = (\mathcal{G}, (\mathcal{G})_{n-1})$ if $n > 1$. Note that $(\mathcal{G}_1, (\mathcal{G}_2, \mathcal{G}_3)) = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = ((\mathcal{G}_1, \mathcal{G}_2), \mathcal{G}_3)$ for any $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ regular. Therefore for \mathcal{G} regular and $m, n \in \mathbb{N}$, $((\mathcal{G})_m, (\mathcal{G})_n) = (\mathcal{G})_{m+n}$. It is easy to see that $\iota((\mathcal{F}, \mathcal{G})) = \iota(\mathcal{G}) + \iota(\mathcal{F})$. We think of $(\mathcal{F}, \mathcal{G})$ as being the sum of \mathcal{F} and \mathcal{G} .

If \mathcal{F}, \mathcal{G} are regular, we let

$$\mathcal{F}[\mathcal{G}] = \left\{ \bigcup_{i=1}^s E_i : (E_i)_{i=1}^s \subset \mathcal{G} \text{ is } \mathcal{F}\text{-admissible} \right\}.$$

This is easily seen to be regular. If \mathcal{G} is regular, we can define $[\mathcal{G}]^n = \mathcal{G}$ if $n = 1$ and $[\mathcal{G}]^n = \mathcal{G}[[\mathcal{G}]^{n-1}]$ if $n > 1$. We observe that $\mathcal{G}_1[\mathcal{G}_2[\mathcal{G}_3]] = (\mathcal{G}_1[\mathcal{G}_2])[\mathcal{G}_3]$ if $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are regular, so that for all regular families \mathcal{G} and all $m, n \in \mathbb{N}$, $[\mathcal{G}]^m[[\mathcal{G}]^n] = [\mathcal{G}]^{m+n}$. We note that $\iota(\mathcal{F}[\mathcal{G}]) = \iota(\mathcal{G})\iota(\mathcal{F})$. We think of \mathcal{F} and \mathcal{G} be the product of \mathcal{F} and \mathcal{G} .

If \mathcal{F}_n is a sequence of regular families with $\iota(\mathcal{F}_n) \uparrow \xi$, then we can construct a diagonalization of the sequence

$$\mathcal{F} = \{E : \exists n \leq n \in \mathcal{F}_n\}.$$

Then \mathcal{F} is regular with $\iota(\mathcal{F}) = \xi$.

We are now in a position to define certain regular families which we will use throughout. We define $\mathcal{A}_n = \{E \in [\mathbb{N}]^{<\omega} : |E| \leq n\}$. We let $\mathcal{S} = \{E : n \leq E \in \mathcal{A}_n\}$. We also define $\mathcal{S}_0 = \mathcal{A}_1$, $\mathcal{S}_{\xi+1} = \mathcal{S}[\mathcal{S}_\xi]$, and if $\xi < \omega_1$ is a limit ordinal with \mathcal{S}_ζ defined for each $\zeta < \xi$, we fix $\xi_n \uparrow \xi$ and let

$$\mathcal{S}_\xi = \{E : \exists n \leq E \in \mathcal{S}_{\xi_n+1}\}.$$

We remark that it is known that, in this case, we can choose the ordinals $\xi_n \uparrow \xi$ so that $\mathcal{S}_{\xi_n+1} \subset \mathcal{S}_{\xi_{n+1}}$ for all $n \in \mathbb{N}$ (this was shown in [10]). We observe that the families resulting from this process are not uniquely determined, since we made choices of $\xi_n \uparrow \xi$ at limit ordinals, but this is unimportant for our purposes as long as we have the containment indicated above. Moreover, it is easy to see that $\iota(\mathcal{S}_\xi) = \omega^\xi$. Also, for convenience, we will let $\mathcal{S}_{\omega_1} = [\mathbb{N}]^{<\omega}$. The families $(\mathcal{S}_\xi)_{\xi < \omega_1}$ are called the *Schreier families*.

2.2. Cantor normal form and Hessenberg sums. We collect here definitions and facts about ordinals which will be used throughout. The following facts can be found in [23].

Proposition 2.4. *Fix $\alpha, \beta, \xi \in \mathbf{Ord}$.*

- (i) *If $\alpha < \omega^\xi$, $\alpha + \omega^\xi = \omega^\xi$.*
- (ii) *If $\alpha, \beta < \omega^\xi$, $\alpha + \beta < \omega^\xi$.*
- (iii) *If $0 < \alpha < \omega^{\omega^\xi}$, $\alpha\omega^{\omega^\xi} = \omega^{\omega^\xi}$.*
- (iv) *If $\alpha, \beta < \omega^{\omega^\xi}$, $\alpha\beta < \omega^{\omega^\xi}$.*
- (v) *If $\alpha < \beta$ and $0 < \xi$, $\xi\alpha < \xi\beta$.*
- (vi) *If $\alpha < \beta$, $\omega^\alpha < \omega^\beta$.*
- (vii) *If $\xi \geq \omega$, then there exists $\zeta \in \mathbf{Ord}$ so that $\xi = \omega^\zeta$ if and only if for every limit ordinal $\lambda < \xi$, $\lambda 2 < \xi$.*

- (viii) The function $\zeta \mapsto \xi + \zeta$ is an order preserving bijection between $[\alpha, \beta)$ and $[\xi + \alpha, \xi + \beta)$.
- (ix) Either $\xi = 0$ or there exist unique $k \in \mathbb{N}$, $\alpha_1 > \dots > \alpha_k$, $\alpha_i \in \mathbf{Ord}$ and $n_i \in \mathbb{N}$ so that $\xi = \omega^{\alpha_1} n_1 + \dots + \omega^{\alpha_k} n_k$. This is called the Cantor normal form of ξ .

We note that $\omega^\zeta > \omega^{\alpha_1} n_1 + \dots + \omega^{\alpha_k} n_k$ if and only if $\zeta > \alpha_1$. Ordinals of the form ω^ξ are called *gamma numbers*, while ordinals of the form ω^{ω^ξ} are called *delta numbers*. Both gamma numbers and delta numbers will be important in this work. The only statement above which is not explicitly stated in Monk is (v), so we discuss it. Suppose $\xi = \omega^{\alpha_1} n_1 + \dots + \omega^{\alpha_k} n_k$ is the Cantor normal form of ξ . First, if $k > 1$, then $\lambda = \omega^{\alpha_1} n_1 + \dots + \omega^{\alpha_{k-1}} n_{k-1} < \xi$ is a limit ordinal and $\lambda 2 > \lambda + \omega^{\alpha_k} n_k = \xi$. If $k = 1$, then $\alpha_k > 0$, or else ξ is finite. If $n_1 > 1$, $\lambda = \omega^{\alpha_1} (n_1 - 1) < \xi$ is a limit and $\lambda 2 = \omega^{\alpha_1} (2n_1 - 2) \geq \xi$. Thus if $\xi \geq \omega$ has the property that for each limit ordinal $\lambda < \xi$, $\lambda 2 < \xi$, ξ is a gamma number.

Note that if $f : S \rightarrow S_0$ is a bijection, $(x_i)_{i=1}^n \mapsto ((f(x_i))_{i=1}^n$ is a bijection between $S^{<\omega}$ and $S_0^{<\omega}$ taking trees (resp. B -trees) to trees (resp. B -trees) of the same order. In particular, for any $\xi \in \mathbf{Ord}$ and $n \in \mathbb{N}$, $\zeta \mapsto \omega^\xi n + \zeta$ maps $[1, \omega^\xi)$ bijectively onto $[\omega^\xi n + 1, \omega^\xi(n+1))$. Moreover, for any $\xi \in \mathbf{Ord}$, $\zeta \mapsto \omega^\xi + \zeta$ takes $[1, \omega^{\xi+1})$ bijectively onto $[\omega^\xi + 1, \omega^\xi + \omega^{\xi+1}) = [\omega^\xi + 1, \omega^{\xi+1})$. We will use this fact to take given B -trees on the same set and transfer them in a natural way to B -trees on disjoint sets while preserving the orders of the trees.

If $\xi, \zeta \in \mathbf{Ord}$, by allowing m_i, n_i to be zero, we can find $\alpha_1 > \dots > \alpha_k$ and $m_i, n_i \in \mathbb{N}$ so that

$$\xi = \omega^{\alpha_1} m_1 + \dots + \omega^{\alpha_k} m_k, \quad \zeta = \omega^{\alpha_1} n_1 + \dots + \omega^{\alpha_k} n_k.$$

Then we define the *Hessenberg sum*, or *natural sum*, of ξ and ζ by

$$\xi \oplus \zeta = \omega^{\alpha_1} (m_1 + n_1) + \dots + \omega^{\alpha_k} (m_k + n_k).$$

We observe that for every $\xi \in \mathbf{Ord}$, the set $\{(\alpha, \beta) : \alpha \oplus \beta = \xi\}$ is finite, and if $\alpha \oplus \beta = \omega^\xi$, either $\alpha = \omega^\xi$ or $\beta = \omega^\xi$. From this, it is easy to see that if $(\alpha_i, \beta_i)_{i \in I}$ is such that $\sup_{i \in I} \alpha_i \oplus \beta_i = \omega^\xi$, then either $\sup_{i \in I} \alpha_i = \omega^\xi$ or $\sup_{i \in I} \beta_i = \omega^\xi$. If it were not so, say $\alpha = \sup_{i \in I} \alpha_i, \beta = \sup_{i \in I} \beta_i < \omega^\xi$, then

$$\sup_{i \in I} \alpha_i \oplus \beta_i \leq \alpha \oplus \beta < \omega^\xi.$$

Proposition 2.5. *Let ξ be a limit ordinal and suppose that for each $\zeta < \xi$, we have ordinals $\xi_{0,\zeta}, \xi_{1,\zeta}$ so that $\xi_{0,\zeta} \oplus \xi_{1,\zeta} = \zeta + 1$. Then there exist a set $A \subset [0, \xi)$, ordinals ξ_0, ξ_1 so that $\xi_0 \oplus \xi_1 = \xi$, and $j \in \{0, 1\}$ so that ξ_j is a limit ordinal and*

$$\sup_{\zeta \in A} \xi_{j,\zeta} = \xi_j \text{ and } \min_{\zeta \in A} \xi_{1-j,\zeta} \geq \xi_{1-j}.$$

Proof. Write ξ in Cantor normal form as

$$\xi = \omega^{\alpha_1} n_1 + \dots + \omega^{\alpha_k} n_k,$$

noting that $\alpha_k > 0$. Let $\xi' = \omega^{\alpha_1} n_1 + \dots + \omega^{\alpha_k} (n_k - 1)$ and $\xi'' = \omega^{\alpha_k}$. For each $\zeta \in [0, \xi'']$,

$$\xi_{0, \xi' + \zeta} \oplus \xi_{1, \xi' + \zeta} = \xi' + \zeta + 1 \in (\xi', \xi),$$

where $\xi' > \zeta + 1$ or $\xi' = 0$. If we denote by $\gamma_{0, \zeta}$ the sum of the terms in the Cantor normal form of $\xi_{0, \xi' + \zeta}$ involving $\omega^{\alpha_1}, \dots, \omega^{\alpha_k}$ and denote by $\delta_{0, \zeta}$ the sum of the remaining, smaller terms, we obtain an expression $\xi_{0, \xi' + \zeta} = \gamma_{0, \zeta} + \delta_{0, \zeta}$. If we perform a similar decomposition with $\xi_{1, \xi' + \zeta} = \gamma_{1, \zeta} + \delta_{1, \zeta}$, we deduce that $\xi' = \gamma_{0, \zeta} \oplus \gamma_{1, \zeta}$ and $\zeta + 1 = \delta_{0, \zeta} \oplus \delta_{1, \zeta}$.

Since $\xi' = \gamma_{0, \zeta} \oplus \gamma_{1, \zeta}$, and since there are only finitely many pairs (γ_0, γ_1) satisfying $\gamma_0 \oplus \gamma_1 = \xi'$, there exists a subset $B \subset [0, \xi'']$ with $\sup B = \xi''$ and ordinals γ_0, γ_1 so that $\gamma_0 \oplus \gamma_1 = \xi'$ and for each $\zeta \in B$, $\gamma_{0, \zeta} = \gamma_0$ and $\gamma_{1, \zeta} = \gamma_1$. Moreover, since $\zeta + 1 = \delta_{0, \zeta} \oplus \delta_{1, \zeta}$ for each $\zeta \in B$, we deduce $\sup_{\zeta \in B} \delta_{0, \zeta} \oplus \delta_{1, \zeta} = \xi''$. But by our remarks above, this means there exists $j \in \{0, 1\}$ so that $\sup_{\zeta \in B} \delta_{j, \zeta} = \xi''$. Let $\xi_j = \gamma_j + \xi''$ and $\xi_{1-j} = \gamma_{1-j}$. We finish by letting $A = \{\xi' + \zeta : \zeta \in B\}$. □

3. COLORING LEMMAS

In this section, we prove several coloring lemmas for trees. The first few can be thought of as coloring segments of trees and searching for subsets of our trees which retain tree structure and have monochromatically colored segments. We have different versions of the same principle because we will apply these arguments to multiple kinds of structures in our Banach spaces (local trees, asymptotic trees, and sequences). The principle behind the first collection of lemmas is that one of our structures of order $\zeta\xi$ can be thought of as a structure of order ξ built from structures of order ζ . For this reason, the first lemmas give product estimates. The remaining lemmas are again two versions of the same principle, but instead of coloring segments of trees, we have maximal members of trees being colored by their predecessors. This principle results in sum estimates.

3.1. Product estimates. Suppose $0 < \zeta \in \mathbf{Ord}$ is fixed. If \mathcal{T} is a B -tree and $f : C(\mathcal{T}) \rightarrow \mathbb{R}$ is a function such that for any order preserving $i : \mathcal{T}_\zeta \rightarrow \mathcal{T}$,

$$\inf\{f \circ i(c) : c \in C(\mathcal{T}_\zeta)\} \leq 0,$$

then we say f is ζ *small*. We say f is *strictly ζ small* if this infimum is a minimum. We note that if $f : C(\mathcal{T}) \rightarrow \mathbb{R}$ is ζ small (resp. strictly ζ small), \mathcal{W} is a B -tree, and $i' : \mathcal{W} \rightarrow \mathcal{T}$ is order preserving, then $f \circ i' : C(\mathcal{W}) \rightarrow \mathbb{R}$ is also ζ small (resp. strictly ζ small). This is because if $i : \mathcal{T}_\zeta \rightarrow \mathcal{W}$ is order preserving, $i' \circ i : \mathcal{T}_\zeta \rightarrow \mathcal{T}$ is order preserving. For $\varepsilon > 0$, there exists $c \in C(\mathcal{T}_\zeta)$ with $f \circ i'(i(c)) = f(i' \circ i(c)) < \varepsilon$ (respectively, $f \circ i'(i(c)) = f(i' \circ i(c)) \leq 0$).

Lemma 3.1. *Fix $0 < \zeta$. Let \mathcal{T} be a B -tree and $f : C(\mathcal{T}) \rightarrow \mathbb{R}$ be ζ small. Then if $o(\mathcal{T}) \geq \zeta\xi$, for any $(\varepsilon_n) \subset (0, 1)$, there exists an order preserving $j : \mathcal{T}_\xi \rightarrow C(\mathcal{T})$ so that for each $t \in \mathcal{T}_\xi$,*

$$f \circ j(t) \leq \varepsilon_{|t|}.$$

Moreover, if f is strictly ζ small, the conclusion holds with $\varepsilon_n = 0$ for all $n \in \mathbb{N}$.

Proof. First, we note that it is sufficient to consider $\mathcal{T} = \mathcal{T}_{\zeta\xi}$. This is because we can fix an order preserving $i : \mathcal{T}_{\zeta\xi} \rightarrow \mathcal{T}$ and let $f' = f \circ i : C(\mathcal{T}_{\zeta\xi}) \rightarrow \mathbb{R}$. Then if we have the result for $\mathcal{T}_{\zeta\xi}$, for $(\varepsilon_n) \subset (0, 1)$, we can find $j' : \mathcal{T}_{\zeta\xi} \rightarrow C(\mathcal{T}_{\zeta\xi})$ order preserving so that $f' \circ j'(t) \leq \varepsilon_{|t|}$. Then with $j : \mathcal{T}_{\zeta\xi} \rightarrow C(\mathcal{T})$ defined by $j(t) = i(j'(t))$, $f \circ j(t) = f \circ i \circ j'(t) = f' \circ j'(t) \leq \varepsilon_{|t|}$. We also note that the last part of the lemma follows from a trivial modification of the first part.

We prove the result on $\mathcal{T}_{\zeta\xi}$ by induction on ξ . The $\xi = 0$ case is trivial. Suppose we have the result for ξ and suppose $f : \mathcal{T}_{\zeta(\xi+1)} \rightarrow \mathbb{R}$ is ζ small. Note that by Proposition 2.1, $o(d^{\zeta\xi}(\mathcal{T}_{\zeta(\xi+1)})) = \zeta$. Fix an order preserving $i' : \mathcal{T}_{\zeta} \rightarrow d^{\zeta\xi}(\mathcal{T}_{\zeta(\xi+1)}) \subset \mathcal{T}_{\zeta(\xi+1)}$. There exists $c_1 \in C(\mathcal{T}_{\zeta})$ so that $f(i'(c_1)) \leq \varepsilon_1$. If $\xi = 0$, we are done with $j((1)) = i'(c_1)$. Assume $\xi > 0$. Let $t_0 = \max i'(c_1) \in d^{\zeta\xi}(\mathcal{T}_{\zeta(\xi+1)})$. Again, by Proposition 2.1, $o(\mathcal{T}_{\zeta(\xi+1)}(t_0)) > \zeta\xi$, whence the B -tree $\mathcal{T}_{\zeta(\xi+1)}(t_0) \setminus \{\emptyset\}$ has order at least $\zeta\xi$. Fix $g : \mathcal{T}_{\zeta\xi} \rightarrow \mathcal{T}_{\zeta(\xi+1)}(t_0) \setminus \{\emptyset\}$ order preserving and define $i'' : \mathcal{T}_{\zeta\xi} \rightarrow \mathcal{T}_{\zeta(\xi+1)}$ by $i''(t) = t_0 \hat{\sim} g(t)$. Note that for each $t \in \mathcal{T}_{\zeta\xi}$, $t_0 \prec i''(t)$, which means that for $c \in C(\mathcal{T}_{\zeta\xi})$, $i'(c_1) < i''(c)$. Note that $f'' = f \circ i'' : C(\mathcal{T}_{\zeta\xi}) \rightarrow \mathbb{R}$ is also ζ small. Let $\varepsilon''_n = \varepsilon_{n+1}$, and note that the inductive hypothesis guarantees the existence of some $j'' : \mathcal{T}_{\zeta\xi} \rightarrow C(\mathcal{T}_{\zeta\xi})$ so that $f'' \circ j''(t) \leq \varepsilon''_n$. Define $j : \mathcal{T}_{\xi+1} \rightarrow C(\mathcal{T}_{\zeta(\xi+1)})$ by $j((\xi+1)) = i'(c_1)$ and for $t \in \mathcal{T}_{\xi}$, $j((\xi+1) \hat{\sim} t) = i'' \circ j''(t)$. Since $i'(c_1) < i''(c)$ for all $c \in C(\mathcal{T}_{\zeta\xi})$, j is order preserving. Moreover $j((\xi+1)) \leq \varepsilon_1 = \varepsilon_{|(\xi+1)|}$ and for $t \in \mathcal{T}_{\xi}$,

$$f \circ j((\xi+1) \hat{\sim} t) = f \circ i'' \circ j''(t) = f'' \circ j''(t) \leq \varepsilon''_{|t|} = \varepsilon_{|(\xi+1) \hat{\sim} t|}.$$

Suppose we have the result for every $\eta < \xi$, ξ a limit ordinal. Assume $f : C(\mathcal{T}_{\zeta\xi}) \rightarrow \mathbb{R}$ is ζ small. Note that for each $\eta < \xi$, $\eta+1 < \xi$, which means $\zeta(\eta+1) < \zeta\xi$. For each $\eta < \xi$, fix an order preserving $i_\eta : \mathcal{T}_{\zeta(\eta+1)} \rightarrow \mathcal{T}_{\zeta\xi}$ and let $f_\eta = f \circ i_\eta : C(\mathcal{T}_{\zeta(\eta+1)}) \rightarrow \mathbb{R}$. Then f_η is ζ small, which means that for any $(\varepsilon_n) \subset (0, 1)$, there exists $j_\eta : \mathcal{T}_{\eta+1} \rightarrow C(\mathcal{T}_{\zeta(\eta+1)})$ so that for each $t \in \mathcal{T}_{\eta+1}$, $f_\eta \circ j_\eta(t) \leq \varepsilon_{|t|}$. Define $j : \mathcal{T}_{\xi} \rightarrow C(\mathcal{T}_{\zeta\xi})$ by letting $j|_{\mathcal{T}_{\eta+1}} = i_\eta \circ j_\eta$. \square

We also state the following strengthening of the previous result which follows from a simple modification of the preceding proof. In what follows, we have a similar scenario as above, except we do not have a single function f , but rather a collection of functions. The difference lies in the fact that we want the order preserving map to choose $j(t)$ to take a small value on functions which are determined by $j(s)$, where $s \prec t$.

Lemma 3.2. *Fix $0 < \zeta \in \mathbf{Ord}$, $\xi \in \mathbf{Ord}$. Suppose that for each $c \in C(\mathcal{T}_{\zeta\xi}) \cup \{\emptyset\}$, $f_c : C(\mathcal{T}_{\zeta\xi}) \rightarrow \mathbb{R}$ is ζ small and for each $c < c'$, $c, c' \in C(\mathcal{T}_{\zeta\xi}) \cup \{\emptyset\}$, and $c_1 \in C(\mathcal{T}_{\zeta\xi})$, $f_c(c_1) \leq f_{c'}(c_1)$. Then for any $(\varepsilon_n) \subset (0, 1)$, there exists an order preserving $j : \mathcal{MT}_{\xi} \rightarrow C(\mathcal{T}_{\zeta\xi}) \cup \{\emptyset\}$ so that $j(\emptyset) = \emptyset$ and so that for each $s \prec t \in \mathcal{T}_{\xi}$,*

$$f_{j(s)}(j(t)) \leq \varepsilon_{|t|}.$$

Moreover, if each f_c is strictly ζ small, the result holds with $\varepsilon_n = 0$ for all $n \in \mathbb{N}$.

The following is an inessential modification of a result from [10]. The previous lemmas were the local version of this principle, while what follows is the asymptotic version.

Lemma 3.3. *Suppose $\emptyset \neq \mathcal{F}, \mathcal{G}$ are regular families. Suppose that for each $c \in C(\mathcal{F}) \cup \{\emptyset\}$, $f_c : \mathcal{F}[\mathcal{G}] \rightarrow \mathbb{R}$ is such that*

- (i) *for $c < c' \in C(\mathcal{F}) \cup \{\emptyset\}$ and any $c_1 \in C(\mathcal{F}[\mathcal{G}])$, $f_c(c_1) \leq f_{c'}(c_1)$,*
- (ii) *for any $E \in \mathcal{F}$ and any embedding $i : \widehat{\mathcal{G}} \rightarrow \mathcal{F}[\mathcal{G}]$,*

$$\inf\{f_E \circ i(c) : c \in C(\mathcal{G})\} = 0.$$

Then for any sequence $(\varepsilon_n) \subset (0, 1)$, there exists an embedding $j : \mathcal{F} \rightarrow C(\mathcal{F}[\mathcal{G}]) \cup \{\emptyset\}$ so that $j(\emptyset) = \emptyset$ and for each $E \prec F \in \mathcal{F}$, $f_{j(E)} \circ j(F) \leq \varepsilon_{|F|}$.

If the infimum above is a minimum, then the conclusion holds with $\varepsilon_n = 0$ for all $n \in \mathbb{N}$.

We also have a mixed version of these results which combines local and asymptotic structures.

Lemma 3.4. *Suppose $0 < \zeta, \xi < \omega_1$. Suppose $f_n : C(\mathcal{T}_{\zeta\xi}) \rightarrow \mathbb{R}$ is a sequence of functions so that*

- (i) *for each $c \in C(\mathcal{T}_{\zeta\xi})$ and each $n \in \mathbb{N}$, $f_n(c) \leq f_{n+1}(c)$,*
- (ii) *for each order preserving $i : \mathcal{T}_\zeta \rightarrow \mathcal{T}_{\zeta\xi}$ and each $n \in \mathbb{N}$,*

$$\inf\{f_n \circ i(c) : c \in C(\mathcal{T}_\zeta)\} = 0.$$

Then for all $\varepsilon_n \downarrow 0$, there exists a regular family \mathcal{F} with $\iota(\mathcal{F}) = \xi$ and an order preserving $j : \widehat{\mathcal{F}} \rightarrow C(\mathcal{T}_{\zeta\xi})$ so that for all $E \in \widehat{\mathcal{F}}$, $f_{\max E}(j(E)) \leq \varepsilon_{\max E}$.

Proof. First, we note that if there exists some regular family \mathcal{F} and an order preserving $j : \widehat{\mathcal{F}} \rightarrow C(\mathcal{T}_{\zeta\xi})$ satisfying the conclusions, then for every regular family \mathcal{G} with $\iota(\mathcal{G}) = \xi$, there exists an order preserving $j' : \widehat{\mathcal{G}} \rightarrow C(\mathcal{T}_{\zeta\xi})$ satisfying the conclusions. This is because if $\iota(\mathcal{G}) = \xi = \iota(\mathcal{F})$, there exists $M \in [\mathbb{N}]$ so that $\mathcal{G}(M) \subset \mathcal{F}$. Then we can define $j' : \widehat{\mathcal{G}} \rightarrow C(\mathcal{T}_{\zeta\xi})$ by $j'(E) = j(M(E))$. Then for $E \in \widehat{\mathcal{G}}$,

$$f_{\max E}(j'(E)) = f_{\max E}(j(M(E))) \leq f_{\max M(E)}(j(M(E))) \leq \varepsilon_{\max M(E)} \leq \varepsilon_{\max E}.$$

Of course, we prove the result again by induction on ξ with ζ held fixed. If $\xi = 1$, letting $i : \mathcal{T}_\zeta \rightarrow \mathcal{T}_{\zeta\xi}$ be the identity, the hypotheses guarantees we can find c_1, c_2, \dots so that $c_n \in C(\mathcal{T}_\zeta)$ and $f_n(c_n) \leq \varepsilon_n$. We define $j : \mathcal{A}_1 \rightarrow C(\mathcal{T}_{\zeta\xi})$ by $j((n)) = c_n$.

Next, suppose the result holds for ξ . We can find an order preserving $i : \mathcal{T}_\zeta \rightarrow d^{\zeta\xi}(\mathcal{T}_{\zeta(\xi+1)})$ and $c_1, c_2, \dots, c_n \in C(\mathcal{T}_\zeta)$ so that $f_n \circ i(c_n) \leq \varepsilon_n$. Letting $t_n = \max c_n$, as in the proof of Lemma 3.1, we find for each n an order preserving $i_n : \mathcal{T}_{\zeta\xi} \rightarrow \mathcal{T}_{\zeta(\xi+1)}(t_n) \setminus \{\emptyset\}$ and use these maps to define functions f_k^n , $k \in \mathbb{N}$, on $\mathcal{T}_{\zeta\xi}$ which satisfy the hypotheses. Then by the inductive hypothesis, for each $n \in \mathbb{N}$, we can find a regular family \mathcal{F}_n with $\iota(\mathcal{F}_n) = \xi$ and an order preserving $j_n : \widehat{\mathcal{F}}_n \rightarrow C(\mathcal{T}_{\zeta\xi}) \rightarrow C(\mathcal{T}_{\zeta(\xi+1)})$, where the second function takes the segment $c \in C(\mathcal{T}_{\zeta\xi})$ to $\{t_n \wedge i_n(t) : t \in c\}$. But by our first remark of the proof, we can

assume that $\mathcal{F}_n = \mathcal{F}_1$ for all $n \in \mathbb{N}$. We let $\mathcal{F} = (\mathcal{A}_1, \mathcal{F}_1)$ and define $j : \widehat{\mathcal{F}} \rightarrow C(\mathcal{T}_{\zeta(\xi+1)})$ by letting $j((n)) = i(c_n)$ and $j(n \smallfrown E) = \{t_n \smallfrown t : t \in j_n(E)\}$. One easily checks that this function satisfies the requirements. Since $\iota(\mathcal{F}) = \iota(\mathcal{F}_1) + \iota(\mathcal{A}_1) = \xi + 1$, this finishes the successor case.

Assume the result holds for each ordinal less than ξ , where ξ is a countable limit ordinal. Choose $\xi_n \uparrow \xi$ arbitrary. For each $n \in \mathbb{N}$, we can define an order preserving $i_n : \mathcal{T}_{\zeta\xi_n} \rightarrow \mathcal{T}_{\zeta\xi}$ and use these maps to define functions $f_k^n : C(\mathcal{T}_{\zeta\xi_n}) \rightarrow \mathbb{R}$, $k \in \mathbb{N}$ which also satisfy the hypotheses. Then for each $n \in \mathbb{N}$, we can find a regular family \mathcal{F}_n with $\iota(\mathcal{F}_n) = \xi_n$ and an order preserving $j_n : \widehat{\mathcal{F}}_n \rightarrow C(\mathcal{T}_{\zeta\xi_n}) \rightarrow C(\mathcal{T}_{\zeta\xi})$ satisfying the requirements. We can recursively choose $M_1 \supset M_2 \supset \dots$, $M_n \in [\mathbb{N}]$ so that with $\mathcal{G}_n = \{E \in [\mathbb{N}]^{<\omega} : M_n(E) \in \mathcal{F}_n\}$, $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \dots$. This is because if \mathcal{F} is any regular family and $M \in [\mathbb{N}]$, $\mathcal{G} = \{E \in [\mathbb{N}]^{<\omega} : M(E) \in \mathcal{F}\}$ is regular with $\iota(\mathcal{F}) = \iota(\mathcal{G})$ [10]. We let $M_1 = \mathbb{N}$ so $\mathcal{G}_1 = \mathcal{F}_1$. Since $\iota(\mathcal{F}_1) < \iota(\mathcal{F}_2)$, we can find $M_2 \in [M_1]$ so that $\mathcal{F}_1(M_2) \subset \mathcal{F}_2$. We then let $\mathcal{G}_2 = \{E \in [\mathbb{N}]^{<\omega} : M_2(E) \in \mathcal{F}_2\}$, so $\mathcal{G}_1 \subset \mathcal{G}_2$. Next, since $\iota(\mathcal{G}_2) < \iota(\mathcal{F}_3)$, we can find $M_3 \in [\mathbb{N}]$ so that $\mathcal{G}_2(M_3) \subset \mathcal{F}_3$. If $\mathcal{G}_3 = \{E \in [\mathbb{N}]^{<\omega} : M_3(E) \in \mathcal{F}_3\}$, $\mathcal{G}_2 \subset \mathcal{G}_3$, and so on. Thus by renaming, we can assume without loss of generality that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$. We let $\mathcal{F} = \{E \in [\mathbb{N}]^{<\omega} : \exists n \leq E \in \mathcal{F}_n\} = \{E \in [\mathbb{N}]^{<\omega} : E \in \mathcal{F}_{\min E}\}$. Note that $\iota(\mathcal{F}) = \sup \xi_n = \xi$. We define $j : \widehat{\mathcal{F}} \rightarrow C(\mathcal{T}_{\zeta\xi})$ by letting $j(E) = j_{\min E}(E)$. This function is easily seen to satisfy the requirements. \square

Finally, we have the simplest version of these results, which we will use on structures determined sequences rather than trees.

Lemma 3.5. *Fix $\xi < \omega_1$. If $f : \widehat{\mathcal{S}}_{\omega^\xi} \rightarrow \{0, 1\}$ is any function, then either there exists $M \in [\mathbb{N}]$ so that for all $E \in \widehat{\mathcal{S}}_{\omega^\xi}$, $f(M(E)) = 1$, or there exists a sequence $E_1 < E_2 < \dots$ so that $f(E_i) = 0$ for all $i \in \mathbb{N}$ and for all $E \in \mathcal{S}_{\omega^\xi}$, $\cup_{i \in \mathbb{N}} E_i \in \mathcal{S}_{\omega^\xi}$.*

Proof. Let $\zeta_n = \xi_n + 1 \uparrow \omega^\xi$ be the sequence used to define \mathcal{S}_{ω^ξ} (we replace \mathcal{S}_{ζ_n} with \mathcal{A}_n in the case that $\xi = 0$). We consider two cases. In the first case, for each $n \in \mathbb{N}$ and for each $N \in [\mathbb{N}]$, there exists $N' \in [N]$ so that for each $n \leq E \in \mathcal{S}_{\zeta_n}$, $f(N'(E)) = 1$. In this case, let $N_0 = \mathbb{N}$ and choose N_1, N_2, \dots so that $N_{n+1} \in [N_n]$ and so that for each $n \leq E \in \mathcal{S}_{\zeta_n}$, $f(N_n(E)) = 1$. Write $N_n = (m_i^n)$ and let $m_n = m_n^n$, $M = (m_n)$. Then M is easily seen to satisfy the conclusions. This is because for $n \leq E \in \mathcal{S}_{\zeta_n}$, $M(E) = N_n(F)$ for some spread F of E . Thus $n \leq F \in \mathcal{S}_{\zeta_n}$ and $f(M(E)) = f(N_n(F)) = 1$.

In the second case, there exist $n \in \mathbb{N}$ and $N \in [\mathbb{N}]$ so that for each $N' \in [N]$, there exists $n \leq E \in \mathcal{S}_{\zeta_n}$ with $f(N'(E)) = 0$. First, choose $L \in [N]$ so that $\mathcal{S}_{\omega^\xi}[\mathcal{S}_{\zeta_n}](L) \subset \mathcal{S}_{\zeta_n + \omega^\xi} = \mathcal{S}_{\omega^\xi}$. Note that for any $s \in \mathbb{N}$, there exists $s \leq F \in \mathcal{S}_{\zeta_n}$ so that $f(L(F)) = 0$. To see this, let $L = (l_k)$ and let $L' = (l'_k) = (l_{s+k}) \in [N]$. By hypothesis, there exists $n \leq E \in \mathcal{S}_{\zeta_n}$ so that $f(L'(E)) = 0$. But

$$L'(E) = (l'_k : k \in E) = (l_{k+s} : k \in E) = (l_k : k \in F) = L(F),$$

where $F = (k + s : k \in E)$. Note that $s \leq F \in \mathcal{S}_{\zeta_n}$, since F is a spread of E . This means we can choose $F_1 < F_2 < \dots$, $F_i \in \mathcal{S}_{\zeta_n}$ so that $f(L(F_i)) = 0$. Let $E_i = L(F_i)$. For $E \in \mathcal{S}_{\omega^\xi}$, $(\min F_i)_{i \in E}$ is a spread of E , so that $(\min F_i)_{i \in E} \in \mathcal{S}_{\omega^\xi}$. This means

$$\bigcup_{i \in E} E_i = L\left(\bigcup_{i \in E} F_i\right) \in \mathcal{S}_{\omega^\xi}[\mathcal{S}_{\zeta_n}](L) \subset \mathcal{S}_{\omega^\xi}.$$

□

3.2. Sum estimates. Given a well-founded B -tree \mathcal{T} and $t \in \mathcal{T}$, we let $\mathcal{E}_t(\mathcal{T}) = \{s \in \text{MAX}(\mathcal{T}) : t \preceq s\}$. We say $(\mathcal{C}_t^0, \mathcal{C}_t^1)_{t \in \mathcal{T}}$ is a *coloring* of \mathcal{T} provided that for each $t \in \mathcal{T}$, $\mathcal{E}_t(\mathcal{T}) = \mathcal{C}_t^0 \cup \mathcal{C}_t^1$. For $j \in \{0, 1\}$, we say a coloring is *monochromatically j* provided that for each $t \in \text{MAX}(\mathcal{T})$,

$$t \in \bigcap_{i=1}^{|t|} \mathcal{C}_{t|_i}^j.$$

For well-founded B -trees \mathcal{T}, \mathcal{W} , we say a pair (i, e) , $i : \mathcal{T} \rightarrow \mathcal{W}$ and $e : \text{MAX}(\mathcal{T}) \rightarrow \text{MAX}(\mathcal{W})$, is an *extended order preserving map* if i is order preserving and for each $t \in \text{MAX}(\mathcal{T})$, $i(t) \preceq e(t)$. We say an extended order preserving map (i, e) is an *extended embedding* if i is an embedding. If $(\mathcal{C}_w^0, \mathcal{C}_w^1)_{w \in \mathcal{W}}$ is a coloring of \mathcal{W} and (i, e) is an extended order preserving map, then

$$\mathcal{D}_t^j = \{s \in \text{MAX}(\mathcal{T}) : e(s) \in \mathcal{C}_{i(j)}^j\}$$

defines a coloring $(\mathcal{D}_t^0, \mathcal{D}_t^1)_{t \in \mathcal{T}}$, which we call the *induced coloring*. Of course, this coloring depends on i , e , and $(\mathcal{C}_w^0, \mathcal{C}_w^1)$, but we will omit reference to the coloring and extended order preserving map inducing the coloring when no confusion will arise. It is clear that if $(\mathcal{C}_w^0, \mathcal{C}_w^1)$ is monochromatically j , then any coloring induced by this coloring is also monochromatically j .

We also note that if $i : \mathcal{T} \rightarrow \mathcal{W}$ is any order preserving map (resp. embedding), and if \mathcal{W} is well-founded, there exists $e : \text{MAX}(\mathcal{T}) \rightarrow \text{MAX}(\mathcal{W})$ so that (i, e) is an extended order preserving map (resp. extended embedding).

Lemma 3.6. *Suppose $0 < \xi \in \text{Ord}$. Suppose $(\mathcal{C}^0, \mathcal{C}^1) \subset \text{MAX}(\mathcal{T}_\xi)$ is such that $\mathcal{C}^0 \cup \mathcal{C}^1 = \text{MAX}(\mathcal{T}_\xi)$. Then there exists an extended order preserving map (i, e) of \mathcal{T}_ξ into \mathcal{T}_ξ and $j \in \{0, 1\}$ so that $e(\mathcal{T}_\xi) \subset \mathcal{C}^j$.*

This lemma should be compared with the result found in [28] which states that if \mathcal{F} is regular, and if we color the maximal members of \mathcal{F} with two colors, there exists $M \in [\mathbb{N}]$ so that $\mathcal{F} \cap [M]^{<\omega}$ is monochromatic. In fact, they prove a stronger result where the set of maximal members of a regular family can be replaced by any family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ so that no member of \mathcal{F} is a subset of another member of \mathcal{F} . Such families are called *thin*.

Lemma 3.7. *If \mathcal{T} is any well-founded B tree on any set, and if $(\mathcal{C}_t^0, \mathcal{C}_t^1)_{t \in \mathcal{T}}$ is any coloring of \mathcal{T} , then for $j \in \{0, 1\}$, there exist an ordinal ξ_j and an extended order preserving map*

(i_j, e_j) of \mathcal{T}_{ξ_j} into \mathcal{T} so that the induced coloring on \mathcal{T}_{ξ_j} is monochromatically j . Moreover, ξ_0, ξ_1 can be chosen so that $\xi_0 \oplus \xi_1 = o(\mathcal{T})$.

Here, it should be understood that if $\xi_j = 0$, we take i_j and e_j to be the empty maps, and that the empty map induces a monochromatically j coloring on the empty set for both $j = 0$ and 1 .

The corresponding result for regular families was shown in [10].

Lemma 3.8. *If \mathcal{F} is a regular family, and if $(\mathcal{C}_E^0, \mathcal{C}_E^1)_{E \in \mathcal{F}}$ is any coloring of \mathcal{F} , then for $j \in \{0, 1\}$, there exist an ordinal $\xi_j < \omega_1$, a regular family \mathcal{F}_j with $\iota(\mathcal{F}_j) = \xi_j$, and an extended embedding (i_j, e_j) of \mathcal{F}_j into \mathcal{F} so that the induced coloring on \mathcal{F}_j is monochromatically j . Moreover, ξ_0, ξ_1 can be chosen so that $\xi_0 \oplus \xi_1 = \iota(\mathcal{F})$.*

Proof of Lemma 3.6. For $\xi = 1$, the result is trivial. Suppose we have the result for ξ . Suppose $\mathcal{C}^0 \cup \mathcal{C}^1 = \text{MAX}(\mathcal{T}_{\xi+1})$. Define $\mathcal{D}^j = \{t \in \mathcal{T}_\xi : (\xi + 1) \hat{\sim} t \in \mathcal{C}^j\}$. Then $\text{MAX}(\mathcal{T}_\xi) = \mathcal{D}^0 \cup \mathcal{D}^1$. Take an extended embedding (i', e') of \mathcal{T}_ξ into \mathcal{T}_ξ and $j \in \{0, 1\}$ so that $e'(\text{MAX}(\mathcal{T}_\xi)) \subset \mathcal{D}^j$. Define $i((\xi + 1)) = (\xi + 1)$, $i((\xi + 1) \hat{\sim} t) = (\xi + 1) \hat{\sim} i'(t)$ for $t \in \mathcal{T}_\xi$, and $e((\xi + 1) \hat{\sim} t) = (\xi + 1) \hat{\sim} e'(t)$ for $t \in \text{MAX}(\mathcal{T}_\xi)$.

Assume we have the result for every $\zeta < \xi$, ξ a limit ordinal. For $\zeta < \xi$ and $j \in \{0, 1\}$, let $\mathcal{C}_\zeta^j = \{t \in \text{MAX}(\mathcal{T}_{\zeta+1}) : t \in \mathcal{C}^j\}$. Find an extended order preserving map (i_ζ, e_ζ) of $\mathcal{T}_{\zeta+1}$ into itself and $j_\zeta \in \{0, 1\}$ so that $e_\zeta(\text{MAX}(\mathcal{T}_{\zeta+1})) \subset \mathcal{C}_\zeta^{j_\zeta}$. For $j \in \{0, 1\}$, let $A_j = \{\zeta < \xi : j_\zeta = j\}$. Since $[0, \xi) = A_0 \cup A_1$, there is $j \in \{0, 1\}$ so that $\sup A_j = \xi$. Thus we can choose $\phi : [0, \xi) \rightarrow A_j$ so that $\zeta < \phi(\zeta)$ for all $\zeta < \xi$. Then for each $\zeta < \xi$, we can find an extended order preserving map (g_ζ, h_ζ) of $\mathcal{T}_{\zeta+1}$ into $\mathcal{T}_{\phi(\zeta)+1}$. Define i on $\mathcal{T}_{\zeta+1}$ and e on $\text{MAX}(\mathcal{T}_{\zeta+1})$ by letting

$$i|_{\mathcal{T}_{\zeta+1}} = i_{\phi(\zeta)} \circ g_\zeta, \quad e|_{\text{MAX}(\mathcal{T}_{\zeta+1})} = e_{\phi(\zeta)} \circ h_\zeta.$$

□

Proof of Lemma 3.7. If $\xi = 1$, the assertion is trivial.

Assume we have the result for an ordinal ξ and suppose first that $\text{MAX}(\mathcal{T}_{\xi+1}) \subset \mathcal{C}_{(\xi+1)}^0$. We then apply the inductive hypothesis to the coloring $(\mathcal{C}_{(\xi+1)}^0 \hat{\sim} t, \mathcal{C}_{(\xi+1)}^1 \hat{\sim} t)_{t \in \mathcal{T}_\xi}$ to deduce the existence of ξ_j , $(i'_j, e'_j) : \mathcal{T}_{\xi_j} \rightarrow \mathcal{T}_\xi \rightarrow \mathcal{T}_{\xi+1}$, where the map from \mathcal{T}_ξ to $\mathcal{T}_{\xi+1}$ is given by $t \mapsto (\xi + 1) \hat{\sim} t$, so that the induced coloring on \mathcal{T}_{ξ_j} is monochromatically j and so that $\xi_0 \oplus \xi_1 = \xi$. We define $(i_1, e_1) : \mathcal{T}_{\xi_1} \rightarrow \mathcal{T}_{\xi+1}$ by simply letting $i_1 = i'_1$ and $e_1 = e'_1$. We define $(i_0, e_0) : \mathcal{T}_{\xi_0+1} \rightarrow \mathcal{T}_{\xi+1}$ by

$$\begin{aligned} i_0((\xi_0 + 1)) &= (\xi + 1), \\ i_0((\xi_0 + 1) \hat{\sim} t) &= i'_0(t), \\ e_0((\xi_0 + 1) \hat{\sim} t) &= e'_0(t). \end{aligned}$$

It is easy to see that these define extended order preserving maps, and that the coloring induced by (i_1, e_1) is monochromatically 1 . To see that the coloring on \mathcal{T}_{ξ_0+1} induced by

(i_0, e_0) is monochromatically 0, take $s \in \text{MAX}(\mathcal{T}_{\xi_0+1})$ and write $s = (\xi_0 + 1) \wedge t$. Then by the properties of (i'_0, e'_0) ,

$$e_0(s) = e_0((\xi_0 + 1) \wedge t) = e'_0(t) \in \bigcap_{k=1}^{|t|} \mathcal{C}_{i'_0(t|_k)}^0 = \bigcap_{k=2}^{|s|} \mathcal{C}_{i_0(s|_k)}^0.$$

But $e_0(s) \in \text{MAX}(\mathcal{T}_{\xi+1}) \subset \mathcal{C}_{(\xi+1)}^0 = \mathcal{C}_{i_0(s|_1)}^0$, so that $e_0(s) \in \bigcap_{k=1}^{|s|} \mathcal{C}_{i_0(s|_k)}^0$. Since $(\xi_0 + 1) \oplus \xi_1 = \xi_0 \oplus \xi_1 + 1 = \xi + 1$, this finishes the proof in this special case. How to complete the proof in the special case that $\text{MAX}(\mathcal{T}_{\xi+1}) \subset \mathcal{C}_{(\xi+1)}^1$ is similar.

For the general successor case, we simply reduce to one of these special cases by Lemma 3.6. With $\mathcal{C}^0 = \mathcal{C}_{(\xi+1)}^0$ and $\mathcal{C}^1 = \mathcal{C}_{(\xi+1)}^1$, we can find an extended order preserving map $(i', e') : \mathcal{T}_{\xi+1} \rightarrow \mathcal{T}_{\xi+1}$ so that $e'(\text{MAX}(\mathcal{T}_{\xi+1})) \subset \mathcal{C}^j$ for either $j = 0$ or $j = 1$. We let $(\mathcal{D}_t^0, \mathcal{D}_t^1)_{t \in \mathcal{T}_{\xi+1}}$ be the coloring induced by (i', e') and $(\mathcal{C}_t^0, \mathcal{C}_t^1)_{t \in \mathcal{T}_{\xi+1}}$. Then the coloring $(\mathcal{D}_t^0, \mathcal{D}_t^1)_{t \in \mathcal{T}_\xi}$ is one of the special cases above. We find ξ_0, ξ_1 , and $(i''_j, e''_j) : \mathcal{T}_{\xi_j} \rightarrow \mathcal{T}_{\xi+1}$ to satisfy the requirements with respect to $(\mathcal{D}_t^0, \mathcal{D}_t^1)_{t \in \mathcal{T}_{\xi+1}}$, and then define $i_j = i' \circ i''_j$, $e_j = e' \circ e''_j$.

Suppose we have the result for every ordinal less than the limit ordinal ξ . If $(\mathcal{C}_t^0, \mathcal{C}_t^1)_{t \in \mathcal{T}_\xi}$ is a coloring, then for each $\zeta < \xi$, $(\mathcal{C}_t^0, \mathcal{C}_t^1)_{t \in \mathcal{T}_{\zeta+1}}$ is a coloring of $\mathcal{T}_{\zeta+1}$. Then we can find $\xi_{j,\zeta}$ and extended order preserving maps $(i_{j,\zeta}, e_{j,\zeta}) : \mathcal{T}_{\xi_{j,\zeta}} \rightarrow \mathcal{T}_{\zeta+1} \rightarrow \mathcal{T}_\xi$ so that the coloring induced on $\mathcal{T}_{\xi_{j,\zeta}}$ is monochromatically j , and so that $\xi_{0,\zeta} \oplus \xi_{1,\zeta} = \zeta + 1$. Then by Proposition 2.5, there exist a set $A \subset [0, \xi)$, ordinals ξ_0, ξ_1 so that $\xi_0 \oplus \xi_1 = \xi$, and $j \in \{0, 1\}$ so that ξ_j is a limit ordinal and

$$\sup_{\zeta \in A} \xi_{j,\zeta} = \xi_j, \quad \min_{\zeta \in A} \xi_{1-j,\zeta} \geq \xi_{1-j}.$$

Without loss of generality, assume $j = 0$.

Fix any $\phi : [0, \xi_0) \rightarrow A$ so that $\xi_{0,\phi(\zeta)} > \zeta + 1$. Then for each $\zeta < \xi_0$, we can define an extended order preserving map $(i'_\zeta, e'_\zeta) : \mathcal{T}_{\zeta+1} \rightarrow \mathcal{T}_{\xi_{0,\phi(\zeta)}}$. Then $(i_{0,\phi(\zeta)} \circ i'_\zeta, e_{0,\phi(\zeta)} \circ e'_\zeta) : \mathcal{T}_{\zeta+1} \rightarrow \mathcal{T}_\xi$ defines an extended order preserving map which is monochromatically 0. We define i_0 and e_0 on \mathcal{T}_{ξ_0} by letting the restriction to $\mathcal{T}_{\zeta+1}$ be these compositions, which defines an extended order preserving $(i_0, e_0) : \mathcal{T}_{\xi_0} \rightarrow \mathcal{T}_\xi$ so that the induced coloring is monochromatically 0.

Choose any $\zeta \in A$ and choose an extended order preserving map $(i', e') : \mathcal{T}_{\xi_1} \rightarrow \mathcal{T}_{\xi_{1,\zeta}}$. Defining $(i_1, e_1) = (i_{1,\zeta} \circ i', e_{1,\zeta} \circ e') : \mathcal{T}_{\xi_1} \rightarrow \mathcal{T}_\xi$ gives an extended order preserving map from $\mathcal{T}_{\xi_1} \rightarrow \mathcal{T}_\xi$ such that the induced coloring is monochromatically 1. Since $\xi_0 \oplus \xi_1 = \xi$, this finishes the proof. □

Of course, we can now apply these results to colorings with any finite number of colors rather than simply two colors. Moreover, if we have any finite coloring $(\mathcal{C}_t^0, \dots, \mathcal{C}_t^n)_{t \in \mathcal{T}_{\omega^\xi}}$, we obtain ξ_0, \dots, ξ_n so that $\xi_0 \oplus \dots \oplus \xi_n = \omega^\xi$ and extended order preserving maps inducing monochromatic colorings. But as we have already discussed, if $\xi_0 \oplus \dots \oplus \xi_n = \omega^\xi$, there exists $0 \leq i \leq n$ so that $\xi_i = \omega^\xi$, thus in this case we obtain a monochromatic structure of

the same size. The same holds with colorings of regular families \mathcal{F} with $\iota(\mathcal{F}) = \omega^\xi$ for some ξ , and in particular for the Schreier families.

We wish to discuss a particular kind of coloring, which is that in which every member t of \mathcal{T}_ξ colors all of the members of $\mathcal{E}_t(\mathcal{T}_\xi)$ with the same color. This is simply the case of a function $f : \mathcal{T}_\xi \rightarrow \{0, \dots, n\}$. In this case, $\mathcal{C}_t^i = \emptyset$ if $i \neq f(t)$ and $\mathcal{C}_t^{f(t)} = \mathcal{E}_t(\mathcal{T}_\xi)$. In this case, the function e plays no real part in the result. That is, if we have $(i_j, e_j) : \mathcal{T}_{\xi_j} \rightarrow \mathcal{T}$ inducing a monochromatically j coloring on \mathcal{T}_{ξ_j} , and if $e'_j : \text{MAX}(\mathcal{T}_{\xi_j}) \rightarrow \text{MAX}(\mathcal{T})$ is any function so that (i_j, e'_j) is also an extended order preserving map, (i_j, e'_j) also induced a monochromatically j coloring.

Corollary 3.9. *If S is any finite set, $\xi \in \mathbf{Ord}$ and if $f : \mathcal{T}_{\omega^\xi} \rightarrow S$ is any function, then there exists an order preserving $i : \mathcal{T}_{\omega^\xi} \rightarrow \mathcal{T}_{\omega^\xi}$ so that $f \circ i$ is constant. The same holds if $\xi < \omega_1$ and if we replace \mathcal{T}_{ω^ξ} with \mathcal{S}_ξ and order preserving with embedding.*

Corollary 3.10. *If $0 < a < b$ and if $f : \mathcal{T}_{\omega^\xi} \rightarrow [a, b]$ is any function, then for any $\delta > 0$, there exists $\theta \in [a, b]$ and an order preserving $i : \mathcal{T}_{\omega^\xi} \rightarrow \mathcal{T}_{\omega^\xi}$ so that for all $t \in \mathcal{T}_{\omega^\xi}$,*

$$\theta \leq f \circ i(t) \leq (1 + \delta)\theta.$$

Proof. Choose $n \in \mathbb{N}$ so that $(b/a)^{1/n} < 1 + \delta$. For $t \in \mathcal{T}_{\omega^\xi}$, let $g(t) = j$ if

$$a(1 + \delta)^{j-1} \leq f(t) < a(1 + \delta)^j.$$

This defines a finite coloring of \mathcal{T}_{ω^ξ} . If $i : \mathcal{T}_{\omega^\xi} \rightarrow \mathcal{T}_{\omega^\xi}$ is order preserving so that $g \circ i \equiv j$, then taking $\theta = a(1 + \delta)^{j-1}$ gives the result. □

4. STRUCTURES IN BANACH SPACES

4.1. Coordinate systems. We let **Ban** denote the class of Banach spaces, and **SB** the class of separable Banach spaces. If X is a Banach space, we let S_X , B_X denote the unit sphere and unit ball of X , respectively. If $A \subset X$, we let $[A]$ denote the closed span of A . If X is a Banach space, a sequence of finite dimensional subspaces $E = (E_n)$ of X is called a *finite dimensional decomposition* (or FDD) for X provided that for each $x \in X$, there exists a unique sequence $(x_n) \subset X$ so that $x_n \in E_n$ for all $n \in \mathbb{N}$ and so that $x = \sum x_n$. In this case, the map defined by $x = \sum x_n \mapsto x_m$ is a well-defined, bounded, linear operator, called the m^{th} *canonical projection* and denoted P_m^E . If $A \in [\mathbb{N}]^{<\omega}$, we let $P_A^E = \sum_{n \in A} P_n^E$. We note that the principle of uniform boundedness implies that the *projection constant of E in X* , defined by $\sup_{m \leq n} \|P_{[m,n]}^E\|$, is finite. If the projection constant of E in X is 1, we say E is *bimonotone* in X . It is known that if E is an FDD for X , one can equivalently renorm X to make E bimonotone in X with respect to the new norm.

If $x \in X$ and E is an FDD for X , we define the *support* of x , denoted $\text{supp}_E(x)$, to be $\{n \in \mathbb{N} : P_n^E x \neq 0\}$. We define the *range* of x to be the smallest interval in \mathbb{N} containing $\text{supp}_E(x)$, and denote the range of x by $\text{ran}_E(x)$. We let $c_{00}(E)$ denote all vectors in X

having finite support. We say a sequence $(x_n) \subset c_{00}(E)$ of non-zero vectors in X is a *block sequence with respect to E* provided $\text{supp}_E(x_n) < \text{supp}_E(x_{n+1})$ for all $n \in \mathbb{N}$. We let $\Sigma(E, X)$ denote the finite block sequences in X with respect to E .

We note that for each $n \in \mathbb{N}$, E_n^* can be embedded into X^* by $(P_n^E)^*$, but this embedding need not be isometric unless E is bimonotone. We will consider E_n^* as a subspace of X^* and let $E^* = (E_n^*)$. We let $[E^*] = [E_n^* : n \in \mathbb{N}]$. We note that E^* is always an FDD for $[E^*]$ with projection constant in $[E^*]$ not exceeding the projection constant of E in X . We say E is *shrinking* if $[E^*] = X^*$. We say E is *boundedly complete* if E^* is a shrinking FDD for $[E^*]$, in which case $X = [E^*]^*$. These facts can be found in [13].

If $(e_i), (f_i)$ are sequences of the same length in (possibly different) Banach spaces, we say (f_i) C -dominates (e_i) , or that (e_i) is C -dominated by (f_i) , if for all $(a_i) \in c_{00}$,

$$\left\| \sum a_i e_i \right\| \leq C \left\| \sum a_i f_i \right\|.$$

We say that two sequences $(e_i), (f_i)$ are C -equivalent if there exist $a, b > 0$ so that $ab \leq C$ and (e_i) a -dominates and is b -dominated by (f_i) . We say the sequence (e_i) is b -*basic* if for all $1 \leq m \leq n$ and $(a_i)_{i=1}^n$,

$$\left\| \sum_{i=1}^m a_i e_i \right\| \leq b \left\| \sum_{i=1}^n a_i e_i \right\|.$$

We say a sequence is *basic* if it is b -basic for some $b \geq 1$, and the smallest b for which (e_i) is b -basic is called the *basis constant* of (e_i) .

Let X be a Banach space and $K \geq 1$. For $1 \leq p \leq \infty$, we let

$$T_p(X, K) = \left\{ (x_i)_{i=1}^n \in B_X^{<\omega} : \forall (a_i)_{i=1}^n \in S_{\ell_p^n}, K^{-1} \leq \left\| \sum_{i=1}^n a_i x_i \right\| \leq 1 \right\}.$$

We note that this is a closed, p -absolutely convex tree on X which is ill-founded if and only if X admits a sequence which is K -equivalent to the unit vector basis of ℓ_p (resp. c_0 if $p = \infty$). If E is an FDD for X , we let $T_p(X, E, K) = T_p(X, K) \cap \Sigma(E, X)$. We note that this tree is not closed, but it is p -absolutely convex. We also define

$$W(X, K) = \left\{ (x_i)_{i=1}^n \in X^{<\omega} : \|x_i\| \geq 1, \forall (a_i)_{i=1}^n \in S_{\ell_\infty^n}, \left\| \sum_{i=1}^n a_i x_i \right\| \leq K \right\},$$

$$W(X, E, K) = W(X, K) \cap \Sigma(E, X).$$

In the case that E has an FDD, we define a second type of derivation on trees T by

$$d_E(T) = \left\{ (x_i)_{i=1}^n \in T : \exists (y_i), i \leq \text{supp}_E(y_i), (x_1, \dots, x_n, y_i) \in T \quad \forall i \in \mathbb{N} \right\}.$$

We let

$$d_E^0(T) = T,$$

$$d_E^{\xi+1}(T) = d_E(d_E^\xi(T)),$$

and if ξ is a limit ordinal, we let

$$d_E^\xi(T) = \bigcap_{\zeta < \xi} d_E^\zeta(T).$$

We let $o_E(T) = \min\{\xi \in \mathbf{Ord} : d_E^\xi(T) = \emptyset\}$. We think of this derivation as being an asymptotic derivation, while the usual derivation is a local one.

We make the following definitions:

- (i) $I_p(X, K) = o(T_p(X, K))$,
- (ii) $I_p(X, E, K) = o(T_p(X, E, K))$,
- (iii) $I_p^a(X, E, K) = o_E(T_p(X, E, K))$,
- (iv) $J(X, K) = o(W(X, K))$,
- (v) $J(X, E, K) = o(W(X, E, K))$,
- (vi) $J^a(X, E, K) = o_E(W(X, E, K))$.

We let $I_p(X) = \sup_{K \geq 1} I_p(X, K)$, and define $I_p(X, E)$, $I_p^a(X, E)$, $J(X)$, $J(X, E)$, and $J^a(X, E)$ similarly. We note that $I_p(\cdot)$ is originally due to Bourgain [7], and is called the Bourgain ℓ_p index of X . The block indices using a basis were defined in [19], and the asymptotic block derivative d_E was defined in [27]. We remark that $I_p(X) > \omega$ if and only if ℓ_p (resp. c_0) is finitely representable in X , and $I_p(X, E) > \omega$ if and only if ℓ_p (resp. c_0) is block finitely representable in E . Also, $I_p^a(X, E) > \omega$ if and only if for each $n \in \mathbb{N}$, ℓ_p^n is in the n^{th} asymptotic structure of X determined by the filter of tail subspaces $[E_n : n \geq m]$, $m \in \mathbb{N}$, of X . For definitions regarding asymptotic structures, we refer the reader to [22].

We remark that there are other coordinate systems of interest, some of which need not be ordered or countable. For example, Markushevich finite dimensional decompositions, unconditional FDDs, or skipped block decompositions. One can define skipped block trees or trees consisting of vectors with finite, disjoint support and formulate many of the results here for such coordinate systems. However, since the difference between I_p and the asymptotic index I_p^a , which has as an analogue the pointwise null index in the case of an unordered coordinate system, can never be very different, we do not explicitly state the results for each possible coordinate system.

We remark that by Proposition 5 of [27], for any $\xi < \omega_1$, and any Banach space X with FDD E , $I_p^a(X, E, K) > \omega^\xi$ if and only if there exists $(x_F)_{F \in \widehat{\mathcal{S}}_\xi} \subset B_X$ so that

- (i) for each $F \in \widehat{\mathcal{S}}_\xi$, $(x_{F|i})_{i=1}^{|F|} \in T_p(X, E, K)$, and
- (ii) for $F \in \mathcal{S}'_\xi$ and $i > F$, $i \leq \text{supp}_E(x_{F \smallfrown i})$.

More generally, if \mathcal{F} is a regular family and $(x_F)_{F \in \widehat{\mathcal{F}}}$ is such that $(x_{F|i})_{i=1}^{|F|} \in T_p(X, E, K)$ for each $F \in \widehat{\mathcal{F}}$ and if $m_i \leq \text{supp}_E(x_{F \smallfrown m_i})$ for each $F \in \mathcal{F}'$, where $(m_i)_{i \in \mathbb{N}} = (m \in \mathbb{N} : F \smallfrown m \in \mathbb{N})$, then $I_p^a(X, E, K) > \iota(\mathcal{F})$. Moreover, there exists a collection called the *fine Schreier families* [27], so that the ι index of the ξ^{th} family is ξ and if $I_p^a(X, E, K) > \xi$, we can find a collection $(x_F)_{F \in \widehat{\mathcal{F}}} \subset X$ satisfying (i) and (ii) above where \mathcal{F} is the ξ^{th} fine Schreier family. Since for any regular \mathcal{F}, \mathcal{G} with $\iota(\mathcal{F}) = \iota(\mathcal{G})$, there exists $M \in [\mathbb{N}]$ so that $\mathcal{F}(M) \subset \mathcal{G}$

and $\mathcal{G}(M) \subset \mathcal{F}$, the exact regular family used as the index set is not important, only its Cantor-Bendixson index. For this reason, we do not discuss the fine Schreier families.

Note that by standard pruning arguments, such as those detailed in [2], if $I_p^a(X, E, K) > \xi$, we can find a regular family \mathcal{F} with $\iota(\mathcal{F}) = \xi$ and a collection $(x_F)_{F \in \widehat{\mathcal{F}}} \subset X$ so that for each $F \in \widehat{\mathcal{F}}$, $(x_{F|i})_{i=1}^{|F|} \in T_p(X, E, K)$, $x_F \in c_{00}(E)$, and so that for $F \in \mathcal{F}'$, $(x_{F \smallfrown i})_{F < i}$ is a block sequence with respect to E . We will call such a collection an *asymptotic block tree*. Note that all of the remarks here apply as well to the index J^a .

We also define higher order ℓ_p and c_0 spreading models. For a regular family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ and $1 \leq p < \infty$, we say (x_n) is a K - $\ell_p^\mathcal{F}$ spreading model provided that for each $E \in \mathcal{F}$, $(x_n)_{n \in E} \in T_p(X, K)$. Note that the usual definition of a K - $\ell_p^\mathcal{F}$ spreading model does not require that the sequence $(x_i)_{i \in E}$ K -dominates and is 1-dominated by the $\ell_p^{|E|}$ basis for each $E \in \mathcal{F}$. Typically one requires the existence of constants $a, b > 0$ with $ab \leq K$ so that for each $E \in \mathcal{F}$, $(x_i)_{i \in E}$ a -dominates and is b -dominated by the $\ell_p^{|E|}$ basis. Our definition is simply the usual definition with a normalization. The notion of K - $c_0^\mathcal{F}$ spreading model is defined similarly. If $\mathcal{F} = \mathcal{S}_\xi$, we write ℓ_p^ξ (resp. c_0^ξ) in place of $\ell_p^{\mathcal{S}_\xi}$ (resp. $c_0^{\mathcal{S}_\xi}$). Note that, with our convention that $\mathcal{S}_{\omega_1} = [\mathbb{N}]^{<\omega}$, (x_n) is a K - $\ell_p^{\omega_1}$ spreading model if and only if it is 1-dominated by and K -dominates the ℓ_p basis.

We also define a notion related to $c_0^\mathcal{F}$ spreading models which will be used later. We say $(x_n) \subset X$ is a K - $c_0^\mathcal{F}$ *special sequence* provided that for each $E \in \mathcal{F}$, $(x_n)_{n \in E} \in W(X, K)$. As with spreading models, we write K - c_0^ξ special sequence in place of K - $c_0^{\mathcal{S}_\xi}$ special sequence. As we discuss below, if \mathcal{F} contains all singletons and $\iota(\mathcal{F}) \geq \omega$, any K - $c_0^\mathcal{F}$ special sequence (x_n) must be seminormalized and weakly null. Therefore some subsequence of (x_n) is 3/2-basic, which means that this subsequence 3-dominates the c_0 basis. Therefore with these assumptions on \mathcal{F} , any K - c_0^ξ special sequence (x_n) admits a subsequence (y_n) so that $(K^{-1}y_n)$ is a $3K$ - $c_0^\mathcal{F}$ spreading model.

We define the following indices associated with containment of higher order spreading models. For $1 \leq p < \infty$ and $K \geq 1$, we let

$$\mathcal{SM}_p(X, K) = \min\{\xi \leq \omega_1 : X \text{ admits no } K\text{-}\ell_p^\xi \text{ spreading model}\},$$

and $\mathcal{SM}_p(X) = \sup_{K \geq 1} \mathcal{SM}_p(X, K)$. For $p = \infty$, we replace ℓ_p spreading models with c_0 spreading models. Similarly, we let $\mathcal{SJ}(X, K)$ be the minimum ordinal ξ not exceeding ω_1 such that X fails to contain a K - c_0^ξ special sequence, and let $\mathcal{SJ}(X) = \sup_{K \geq 1} \mathcal{SJ}(X, K)$. Note that by our conventions, $\mathcal{SM}_p(X, K) = \infty$ if and only if X admits a sequence K -equivalent to the ℓ_p (resp. c_0) basis.

We make a few simple observations concerning higher order spreading models. Note that if \mathcal{F}, \mathcal{G} are regular families and if $(x_n) \subset X$ is a K - $\ell_p^{\mathcal{F}[\mathcal{G}]}$ spreading model, then any p -absolutely convex block (y_n) of (x_n) with $y_n = \sum_{j \in E_n} a_j x_j$, $E_n \in \mathcal{G}$ for all $n \in \mathbb{N}$, must be a K - $\ell_p^\mathcal{F}$ spreading model.

If $\iota(\mathcal{F}) \geq \omega$, then \mathcal{F} contains sets of arbitrarily large cardinality. Since \mathcal{F} is hereditary, we can fix $E_1 < E_2 < \dots$, $E_n \in \mathcal{F}$ with $|E_n| = n$. If $(x_n) \subset X$ is a $K\text{-}\ell_p^\mathcal{F}$ spreading model for $1 < p \leq \infty$, and if $M \in [\mathbb{N}]$, then

$$\left\| \frac{1}{n} \sum_{i \in E_n} x_{m_i} \right\| \leq n^{1/p}/n \rightarrow 0,$$

whence (x_n) must be a weakly null sequence. Since (x_n) must have a seminormalized subsequence (and must be seminormalized itself if \mathcal{F} contains all singletons), some subsequence of (x_n) is a $(1 + \varepsilon)$ -basic $K\text{-}\ell_p^\mathcal{F}$ spreading model. Moreover, if X has an FDD and admits a $K\text{-}\ell_p^\mathcal{F}$ spreading model for some $K \geq 1$, $1 < p \leq \infty$, and \mathcal{F} with $\iota(\mathcal{F}) \geq \omega$, then X admits a $(K + \varepsilon)\text{-}\ell_p^\mathcal{F}$ spreading model which is a block sequence with respect to the FDD.

Suppose $(x_n) \subset X$ is a $K\text{-}\ell_1^\mathcal{F}$ spreading model. By Rosenthal's ℓ_1 theorem [30], either some subsequence of (x_n) is equivalent to the ℓ_1 basis, or some subsequence is weakly Cauchy. Assume (x_n) itself is weakly Cauchy. If $\iota(\mathcal{F})$ is a limit ordinal, $\iota(\mathcal{F}[\mathcal{A}_2]) = 2\iota(\mathcal{F}) = \iota(\mathcal{F})$, and there must exist $M \in [\mathbb{N}]$ so that $\mathcal{F}[\mathcal{A}_2](M) \subset \mathcal{F}$. Then with $M = (m_n)$, (x_{m_n}) is a $K\text{-}\ell_1^{\mathcal{F}[\mathcal{A}_2]}$ spreading model and therefore $2y_n = x_{m_{2n}} - x_{m_{2n-1}}$ is such that (y_n) is a weakly null $K\text{-}\ell_1^\mathcal{F}$ spreading model. Thus for any $1 \leq \xi < \omega_1$, if X admits a $K\text{-}\ell_1^\xi$ spreading model, then either every $K\text{-}\ell_1^\xi$ spreading model in X admits a subsequence equivalent to the ℓ_1 basis, or X admits a weakly null $K\text{-}\ell_1^\xi$ spreading model.

We last recall the definition of an *asymptotic* $\ell_p^\mathcal{F}$ or $c_0^\mathcal{F}$ basis. Given a basis E for a Banach space X , we say E is K -asymptotic $\ell_p^\mathcal{F}$ (resp. $c_0^\mathcal{F}$ when $p = \infty$) provided that each \mathcal{F} -admissible, normalized block sequence $(x_i)_{i=1}^n$ with respect to E , and for each $(a_i)_{i=1}^n \in S_{\ell_p^n}$,

$$K^{-1} \leq \left\| \sum_{i=1}^n a_i x_i \right\| \leq K.$$

We say a block sequence $(x_i)_{i=1}^n$ is \mathcal{F} -admissible with respect to E provided $(\text{supp}_E(x_i))_{i=1}^n$ is \mathcal{F} -admissible. Note that if E is a K -asymptotic $\ell_1^\mathcal{F}$ basis, then every normalized block sequence with respect to E is a $K\text{-}\ell_1^\mathcal{F}$ spreading model. Note that if E is a bimonotone, K -asymptotic $c_0^\mathcal{F}$ basis, then every normalized block sequence is a $K\text{-}c_0^\mathcal{F}$ special sequence.

It is clear that for a Banach space X with FDD E and $1 \leq p \leq \infty$, each of the properties below is implied by the successive properties, with the replacement of ℓ_p by c_0 in the case that $p = \infty$.

- (i) ℓ_p is finitely representable in X .
- (ii) ℓ_p is finitely representable in X on disjoint blocks of E .
- (iii) ℓ_p is finitely block representable in E .
- (iv) $I_p^a(X, E) > \omega$.
- (v) X admits $(1 + \varepsilon)\text{-}\ell_p^{A_n}$ spreading models for all $\varepsilon > 0$ and all $n \in \mathbb{N}$.

We recall some simple examples to show that no property on the list above implies any of the properties after it, so that the different indices are indeed distinct.

- (i) Every Banach space is finitely representable in c_0 , but the basis of ℓ_p is not finitely representable on disjoint blocks of the c_0 basis.
- (ii) The c_0 basis is finitely representable on disjoint blocks of the basis of Schlumprecht space S [11], which means ℓ_1 is finitely representable on disjoint blocks of the basis of S^* . Therefore ℓ_p is finitely representable on disjoint blocks of the basis of the p -convexification of S^* . But only c_0 is finitely block representable on this basis.
- (iii) The space $X = (\sum \ell_p^n)_{\ell_q}$, where $\infty \neq q \neq p$, has a natural basis E in which ℓ_p is block finitely representable, but it is clear that $I_p^a(X, E) = \omega$. This is because in each asymptotic block tree, some branch will be closely equivalent to the basis of ℓ_q^n for some $n \in \mathbb{N}$.
- (iv) For $1 < q < \infty$ and $1 \leq p < q$, we define for each $n \in \mathbb{N}$ a norm on the finitely supported, scalar-valued functions defined on $\widehat{\mathcal{A}}_n$. We let

$$\|f\|_n = \sup \left\{ \left(\sum_{c \in I} \left(\sum_{F \in c} |f(F)|^p \right)^{q/p} \right)^{1/p} : I \text{ is a collection of pairwise disjoint segments in } \widehat{\mathcal{A}}_n \right\}.$$

We let X_n denote the completion of this set with respect to this norm and note that with $e_F^n = 1_F$ for $F \in \widehat{\mathcal{A}}_n$, $(e_F^n)_{F \in \widehat{\mathcal{A}}_n}$ is a normalized, 1-unconditional basis for X_n . Moreover, each X_n is isomorphic to ℓ_q , and the natural basis is equivalent to the ℓ_q basis. Then we let $X = (\oplus X_n)_{\ell_q}$. Then each basis $(e_F^n)_{F \in \widehat{\mathcal{A}}_n}$ naturally witnesses the fact that $I_p^a(X, E, 1) > n$. But as was shown in [26], every normalized, weakly null sequence in this space has a subsequence $(1 + \varepsilon)$ -equivalent to the unit vector basis of ℓ_q . Since X contains no copy of ℓ_1 , if (x_i) is a K - $\ell_p^{A_{2n}}$ spreading model, we can pass to a weakly Cauchy subsequence and then to the p -absolutely convex block $(2^{-1/p}(x_{2i} - x_{2i-1}))$ to obtain a weakly null K - $\ell_p^{A_n}$ spreading model. By passing to a further subsequence, we can obtain a sequence closely equivalent to the ℓ_q basis, which implies a lower estimate on K as a function of n which tends to infinity as n does. Thus X cannot contain uniform $\ell_p^{A_n}$ spreading models for all $n \in \mathbb{N}$. Moreover, if $p = 1$, the dual of this space satisfies $I_\infty^a(X^*, E^*, 1) > \omega$, while X^* does not admit uniform $c_0^{A_n}$ spreading models.

Next we collect some important facts about these indices to be used throughout. For $p = 1$, a number of the facts below can be found in [19]. Because our goal is precise quantification, we include stronger quantifications than those established in [19]. It is clear that if X is finite dimensional, $I_p(X) = 1 + \dim X$, and otherwise $I_p(X) \geq \omega$. Moreover, if X has an FDD E , $I_p(X, E), I_p^a(X, E) \geq \omega$. We collect the following facts concerning these indices for the infinite dimensional case.

Proposition 4.1. *Let X be an infinite dimensional Banach space. Let $1 \leq p \leq \infty$, $1 \leq K$, and let λ be a limit ordinal.*

- (i) *If $F \subset X^*$ is finite and $Y = \cap_{x^* \in F} \ker(x^*) = F_\perp$, $I_p(X, K) > \lambda$ if and only if $I_p(Y, K) > \lambda$*

- (ii) If E is an FDD for X and if $Y = [E_n : n \geq m]$ and $F = (E_n)_{n \geq m}$, $I_p(X, E, K) > \lambda$ if and only if $I_p(Y, F, K) > \lambda$.
- (iii) If $Y = [E_n : n \geq m]$ and $F = (E_n)_{n \geq m}$, $I_p^a(X, E, K) = I_p^a(Y, F, K)$.
- (iv) Either $I_p(X) = \infty$ or there exists ξ so that $I_p(X) = \omega^\xi$.
- (v) If E is an FDD for X , either $I_p(X) = \infty$ or there exist $\alpha \leq \beta \leq \gamma \leq 1 + \alpha$ so that $I_p^a(X, E) = \omega^\alpha$, $I_p(X, E) = \omega^\beta$, and $I_p(X) = \omega^\gamma$.
- (vi) For any $K \geq 1$, $I_p(X, K) < I_p(X)$ and, if E is an FDD for X , $I_p(X, E, K) < I_p(X, E)$ and $I_p^a(X, E, K) < I_p^a(X, E)$.

Proof. (i) Clearly $I_p(X, K) \geq I_p(Y, K)$. If $I_p(X, K) > \lambda$, fix $(x_t)_{t \in \mathcal{T}_\lambda}$ so that $(x_{t|_i})_{i=1}^{|t|} \in T_p(X, K)$ for all $t \in \mathcal{T}_\lambda$ and fix $m > |F|$. Define a function $f : C(\mathcal{T}_\lambda) = C(\mathcal{T}_{m\lambda}) \rightarrow \mathbb{R}$ by

$$f(c) = \min \left\{ \sum_{x^* \in F} |x^*(x)| : x \in \text{co}_p(x_t : t \in c) \right\}.$$

Then by simply comparing dimensions, for every order preserving $i : \mathcal{T}_m \rightarrow \mathcal{T}_{m\lambda}$, $\min\{f \circ i(c) : c \in C(\mathcal{T}_m)\} = 0$. Thus by 3.1 we can find an order preserving $j : \mathcal{T}_\lambda \rightarrow C(\mathcal{T}_\lambda)$ so that $f \circ j = 0$. For each $t \in \mathcal{T}_\lambda$, fix $u_t \in \text{co}_p(x_s : s \in j(t))$ with $f(u_t) = 0$, which means $u_t \in Y$. Then since j is order preserving, $(u_t)_{t \in \mathcal{T}_\lambda}$ is such that $(u_{t|_i})_{i=1}^{|t|} \in T_p(Y, K)$, since this sequence lies in Y and is a p -absolutely convex block of a member of $T_p(X, K)$. Thus $(u_t)_{t \in \mathcal{T}_\lambda}$ witnesses the fact that $I_p(Y, K) > \lambda$.

(ii) Suppose that $I_p(X, E, K) > \lambda$ and fix $(x_t)_{t \in \mathcal{T}_\lambda}$ so that $(x_{t|_i})_{i=1}^{|t|} \in T_p(X, E, K)$ for each $t \in \mathcal{T}_\lambda$. Let

$$f(c) = \min \{ \|P_{[1,m]}^E x\| : x \in \text{co}_p(x_t : t \in c) \}.$$

We finish as in (i).

(iii) If $I_p^a(X, E, K) > \lambda$, then there exists a regular family \mathcal{F} with $\iota(\mathcal{F}) = \lambda$ and a collection $(x_G)_{G \in \widehat{\mathcal{F}}}$ so that for each $G \in \widehat{\mathcal{F}}$, $(x_{G|_i})_{i=1}^{|G|} \in T_p(X, E, K)$ and $\max G \leq \text{supp}_E(x_G)$. Then with $\mathcal{G} = \mathcal{F} \cap [m, \infty)^{<\omega}$, $(x_G)_{G \in \widehat{\mathcal{G}}}$ witnesses the fact that $I_p^a(Y, F, K) > \lambda$.

(iv) Assume $I_p(X) < \infty$, which means $I_p(X)$ is an infinite ordinal. It is sufficient to show that if $\lambda < I_p(X)$, $\lambda 2 < I_p(X)$. Assume $\lambda < I_p(X, K)$. Choose any sequence $(x_i)_{i=1}^m \in T_p(X, K)$. Fix $\varepsilon > 0$ and choose $F \subset X^*$ finite which is $(1 + \varepsilon)$ -norming for $[x_i : 1 \leq i \leq m]$. Note that in the space $[x_i : 1 \leq i \leq m] \oplus F_\perp$, $[x_i : 1 \leq i \leq m]$ is $(1 + \varepsilon)$ -complemented, and so F_\perp is $(2 + \varepsilon)$ -complemented. If $(u_i)_{i=1}^n \in T_p(F_\perp, K)$, then for any scalars $(a_i)_{i=1}^m$ and $(b_i)_{i=1}^n$,

$$\begin{aligned} \left\| \sum_{i=1}^m a_i x_i + \sum_{i=1}^n b_i u_i \right\| &\leq \left\| \sum_{i=1}^m a_i x_i \right\| + \left\| \sum_{i=1}^n b_i u_i \right\| \leq \left(\sum_{i=1}^m |a_i|^p \right)^{1/p} + \left(\sum_{i=1}^n |b_i|^p \right)^{1/p} \\ &\leq 2^{1/q} \left(\sum_{i=1}^m |a_i|^p + \sum_{i=1}^n |b_i|^p \right)^{1/p}, \end{aligned}$$

where q is the conjugate exponent to p . Moreover,

$$\begin{aligned} \left\| \sum_{i=1}^m a_i x_i + \sum_{i=1}^n b_i u_i \right\| &\geq (2 + \varepsilon)^{-1} \max \left\{ \left\| \sum_{i=1}^m a_i x_i \right\|, \left\| \sum_{i=1}^n b_i u_i \right\| \right\} \\ &\geq (2 + \varepsilon)^{-1} / K \max \left\{ \left(\sum_{i=1}^m |a_i|^p \right)^{1/p}, \left(\sum_{i=1}^n |b_i|^p \right)^{1/p} \right\} \\ &\geq 2^{-1/p} (2 + \varepsilon)^{-1} / K. \end{aligned}$$

Therefore $2^{-1/q}(x_1, \dots, x_m, u_1, \dots, u_n) \in T_p(X, 2(2 + \varepsilon)K)$. But since this holds for any $(u_1, \dots, u_n) \in T_p(F_\perp, K)$, and since $I_p(F_\perp, K) > \lambda$,

$$\left\{ 2^{-1/q}(x_1, \dots, x_m) : (x_1, \dots, x_m) \in T_p(X, K) \right\} \subset d^\lambda(T_p(X, 2(2 + \varepsilon)K)).$$

Since the tree on the left also has order exceeding λ , we deduce that for any $\delta > 0$, $I_p(X, 4K + \delta) > \lambda 2$. It is easy to see how to modify this proof to see that if $I_p(X, K) > \lambda$, then for any $n \in \mathbb{N}$ and $\varepsilon > 0$, $I_p(X, 2nK + \varepsilon) > \lambda n$. We simply select $(x_i^1)_{i=1}^{l_1} \in T_p(X, K)$, $F^1 \subset X^*$ finite and $(1 + \varepsilon)$ -norming for $[x_i^1 : 1 \leq i \leq l_1]$. Then we choose $(x_i^2)_{i=1}^{l_2} \in T_p(F_\perp^1, K)$ and $F^2 \subset X^*$ finite and $(1 + \varepsilon)$ -norming for $[x_i^2 : 1 \leq i \leq l_2]$. Choose $(x_i^3)_{i=1}^{l_3} \in T_p(F_\perp^2, K)$, and so on. Then

$$\left\| \sum_{j=1}^n \sum_{i=1}^{l_j} a_{ij} x_i^j \right\| \leq n^{1/q} \left(\sum_{j=1}^n \sum_{i=1}^{l_j} |a_{ij}|^p \right)^{1/p}$$

and

$$\begin{aligned} \left\| \sum_{j=1}^n \sum_{i=1}^{l_j} a_{ij} x_i^j \right\| &\geq K^{-1} (1 + \varepsilon)^{-1} (2 + \varepsilon)^{-1} \max_{1 \leq j \leq n} \left\{ \left(\sum_{i=1}^{l_j} |a_{ij}|^p \right)^{1/p} \right\} \\ &\geq n^{-1/p} K^{-1} (1 + \varepsilon)^{-1} (2 + \varepsilon)^{-1} \left(\sum_{j=1}^n \sum_{i=1}^{l_j} |a_{ij}|^p \right)^{1/p}. \end{aligned}$$

(v) Again, assume $I_p(X) < \infty$. The proof of the existence of α and β is essentially the same as the previous case, except we assume that $(x_i)_{i=1}^m$ is a block sequence so that $\text{supp}_E(x_m) \subset [1, k)$ for some k , and then concatenate this with members of $T_p([E_n : n \geq k], K)$. We note that in the case that E is bimonotone, if $(x_i^1)_{i=1}^{l_1}, \dots, (x_i^n)_{i=1}^{l_n}$ are block sequences so that the concatenation is also a block sequence, then

$$\left\| \sum_{j=1}^n \sum_{i=1}^{l_j} a_{ij} x_i^j \right\| \geq \max_{1 \leq j \leq n} \left\{ \left\| \sum_{i=1}^{l_j} a_{ij} x_i^j \right\| \right\},$$

so that in this case we have that if $\lambda < I_p(X, E, K)$, then $\lambda n < I_p(X, E, Kn)$. We will prove later that this lower estimate on the growth rate of $I_p(X, E, \cdot)$ is sharp.

Note that at this point, we have established that $I_p(X)$, $I_p(X, E)$, and $I_p^a(X, E)$ are limit ordinals, which gives (vi). We will use this to prove that $\gamma \leq 1 + \alpha$. If $\gamma > 1 + \alpha$, then $\omega^\gamma > \omega^{1+\alpha} = \omega I_p^a(X, E) > \omega I_p^a(X, E, K)$ for all K . But this means there exists some

$K \geq 1$ so that $I_p(X, K) > \omega I_p^a(X, E, K)$. Let $\xi = I_p^a(X, E, K)$ and let $(x_t)_{t \in \mathcal{T}_{\omega\xi}}$ be so that $(x_{t_i})_{i=1}^{|t|} \in T_p(X, K)$ for each $t \in \mathcal{T}_{\omega\xi}$. By replacing K with any strictly larger value, we can assume that each x_t has finite support. For each $n \in \mathbb{N}$, define $f_n : C(\mathcal{T}_{\omega\xi}) \rightarrow \mathbb{R}$ by

$$f_n(c) = \left\{ \sum_{i=1}^n \|P_{[1,i]}^E x\| : x \in \text{co}_p(x_t : t \in c) \right\}.$$

Again, simply by comparing dimensions, for any $n \in \mathbb{N}$ and any order preserving $i : \mathcal{T}_{\omega} \rightarrow \mathcal{T}_{\omega\xi}$,

$$\min\{f_n \circ i(c) : c \in C(\mathcal{T}_{\omega})\} = 0.$$

Note that $f_1 \leq f_2 \leq f_3 \leq \dots$ pointwise. Thus by Lemma 3.4 we can find a regular family \mathcal{F} with $\iota(\mathcal{F}) = \xi$ and an order preserving $j : \widehat{\mathcal{F}} \rightarrow C(\mathcal{T}_{\omega\xi})$ so that for each $G \in \widehat{\mathcal{F}}$,

$$f_{\max G}(j(G)) = 0.$$

Choose for each $G \in \widehat{\mathcal{F}}$ some $u_G \in \text{co}_p(x_t : t \in j(G))$ so that $\sum_{i=1}^n \|P_{[1,i]} u_G\| = 0$, which means $\max G \leq \text{supp}_E(x_G)$. Then $(u_G)_{G \in \widehat{\mathcal{F}}}$ witnesses the fact that $I_p^a(X, K) > \xi$, a contradiction. □

Next, we define for each ordinal a class of Banach spaces admitting (i) a crude ℓ_p structure of a certain size, and (ii) almost isometric ℓ_p structures of a certain size. These classes will be our primary objects of study. For $1 \leq p \leq \infty$ and $\xi \in \mathbf{Ord}$, we let

$$P_1^\vee(\xi, p) = \bigcup_{K > 1} \{X \in \mathbf{Ban} : I_p(X, K) > \omega^\xi\},$$

$$P_1^\wedge(\xi, p) = \bigcap_{K > 1} \{X \in \mathbf{Ban} : I_p(X, K) > \omega^\xi\},$$

$$P_2^\vee(\xi, p) = \bigcup_{K > 1} \{(E, X) : E \text{ is an FDD for } X \in \mathbf{SB}, I_p(X, E, K) > \omega^\xi\},$$

$$P_2^\wedge(\xi, p) = \bigcap_{K > 1} \{(E, X) : E \text{ is an FDD for } X \in \mathbf{SB}, I_p(X, E, K) > \omega^\xi\}.$$

$$P_3^\vee(\xi, p) = \bigcup_{K > 1} \{(E, X) : E \text{ is an FDD for } X \in \mathbf{SB}, I_p^a(X, E, K) > \omega^\xi\},$$

$$P_3^\wedge(\xi, p) = \bigcap_{K > 1} \{(E, X) : E \text{ is an FDD for } X \in \mathbf{SB}, I_p^a(X, E, K) > \omega^\xi\},$$

$$P_4^\vee(\xi, p) = \bigcup_{K > 1} \{X \in \mathbf{Ban} : \mathcal{SM}_p(X, K) > \xi\},$$

$$P_4^\wedge(\xi, p) = \bigcap_{K > 1} \{X \in \mathbf{Ban} : \mathcal{SM}_p(X, K) > \xi\}.$$

Of course, it is well known that $P_j^\vee(1, p) = P_j^\wedge(1, p)$ for all $1 \leq p \leq \infty$. This is a result of James for $p = 1$ and $p = \infty$ [18], and a consequence of Krivine's theorem [20] for $1 < p < \infty$. In sections 6 and 7, we discuss when $P_i^\vee(\xi, p) = P_i^\wedge(\xi, p)$ for $i = 1, 2, 3, 4$, $1 \leq p \leq \infty$, and $\xi \in \mathbf{Ord}$. In particular, for $i = 2, 3, 4$, we completely solve this question for all ordinals when

$1 < p < \infty$, and for all countable ordinals when $p = 1$ or ∞ . Typically the study of these indices has been restricted to the case of separable spaces and countable indices. We first give a simple construction that shows that for any $\xi \in \mathbf{Ord}$ and for any $1 \leq p \leq \infty$, there exists X with $I_p(X) > \xi$ and not containing a copy of ℓ_p .

Proposition 4.2. *For any $\xi \in \mathbf{Ord}$ and any $1 \leq p \leq \infty$, there exists a Banach space $U_{\xi,p}$ with $\omega^\xi < I_p(U_{\xi,p}, 1)$ containing no copy of ℓ_p (resp. c_0). More precisely, we can choose $U_{\xi,1}$ to be reflexive with 1-unconditional (not necessarily countable) basis E_ξ so that $\omega^\xi < I_1(U_{\xi,1}, E_\xi, 1)$, $U_{\xi,\infty} = U_{\xi,1}^*$, and $U_{\xi,p}$ is the p -convexification of $U_{\xi,1}$.*

Here, since the basis E_ξ is unordered, we interpret $T_p(U_{\xi,1}, E_\xi, 1)$ as being the collection of finite, disjointly supported sequences isometrically equivalent to the ℓ_p^n basis, and $I_p(U_{\xi,1}, E_\xi, 1)$ is the order of this tree.

Proof. First, note that for any $\xi, \zeta \in \mathbf{Ord}$, any Banach spaces X, Y , any $K \geq 1$, and $1 \leq p \leq \infty$, if $\xi < I_p(X, K)$ and $\zeta < I_p(Y, K)$, $\zeta + \xi < I_p(X \oplus_p Y, K)$. This is because for any $s \in T_p(X, K)$, $T_p(Y, K) \subset T_p(X \oplus_p Y, K)(s)$, so that $s \in d^\zeta(T_p(X \oplus_p Y, K))$. This means $T_p(X, K) \subset d^\zeta(T_p(X \oplus_p Y, K))$, and $\emptyset \neq d^\xi(T_p(X, K)) \subset d^{\zeta+\xi}(T_p(X \oplus_p Y, K))$. Similarly, if E is an unconditional basis for X and F is an unconditional basis for Y , $I_p(X \oplus_p Y, E \cup F, K) > \zeta + \xi$ if $\xi < I_p(X, E, K)$ and $\zeta < I_p(Y, F, K)$.

It is clear that if E_ξ is a 1-unconditional basis for a reflexive $U_{\xi,1}$ and $\omega^\xi < I_1(U_{\xi,1}, E_\xi, 1)$, then the p -convexification $U_{\xi,1}^{(p)}$ with the same basis E_ξ will contain no copy of ℓ_p and satisfy $\omega^\xi < I_p(U_{\xi,1}^{(p)}, E_\xi, 1)$. Thus it is sufficient to treat the $p = 1$ case and its dual.

First, we take $U_{0,1} = \mathbb{R}$ in the real case, $U_{0,1} = \mathbb{C}$ in the complex case. Therefore $U_{0,1}^* = U_{0,1}$ and $I_1(U_{0,1}, E_0, 1) = I_\infty(U_{0,1}, E_0, 1) = 2$.

If $U_{\xi,1}$ has been defined, we let $V_1 = U_{\xi,1}$ and $V_{n+1} = U_{\xi,1} \oplus_1 V_n$. Then V_n is reflexive with 1-unconditional basis, say F_n , and $V_n^* = (\oplus U_{\xi,1}^*)_{\ell_\infty^n}$. Moreover, for each $n \in \mathbb{N}$, since $I_1(U_{\xi,1}, E_\xi, 1) > \omega^\xi$, $I_1(V_n, F_n, 1) > \omega^\xi n$. Similarly, if F_n^* is the dual basis to F_n , $I_\infty(V_n^*, F_n^*, 1) > \omega^\xi n$. We take $U_{\xi+1,1} = (\oplus V_n)_{\ell_2}$.

Last, suppose ξ is a limit ordinal, and that for each $\zeta < \xi$, $U_{\zeta,1}$ has been defined to have the announced properties. We take $U_{\xi,1} = (\oplus U_{\zeta,1})_{\ell_2([1,\xi])}$.

□

It is not hard to see that these spaces do not admit ℓ_1^1 spreading models. We will later see that if W is any non-zero Banach space not admitting an $\ell_1^{\omega^\zeta}$ spreading model, if we repeat the above construction with W in place of $U_{0,1}$, we build up a collection $W_{\xi,1}$ so that $I_1(W_{\xi,1}, 1) > \omega^\xi$ admitting no $\ell_1^{\omega^\zeta}$ spreading model. Of course, these spaces contain ℓ_2 for $\xi > 0$. In [5], remarkable examples were given of Banach spaces W_ξ , $\xi < \omega_1$, so that $I_1(X) > \xi$ for every infinite dimensional subspace X of W_ξ , while W_ξ admits no ℓ_1^1 spreading model.

5. DUALIZATION

5.1. Bourgain-Delbaen constructions. It was asked in [9] if whenever a Banach space X contains a copy of ℓ_p , $1 < p < \infty$, must X^* contain a copy of ℓ_q , where q is the conjugate exponent to p ? It was shown in [8] that there exists an \mathcal{L}_∞ Banach space Z with dual isomorphic to ℓ_1 so that every infinite dimensional subspace of Z admits a further infinite dimensional reflexive subspace. Haydon [17] actually showed that for any $1 < p < \infty$, there exists an \mathcal{L}_∞ Banach space Z_p so that Z_p^* is isomorphic to ℓ_1 and so that ℓ_p embeds into Z_p . Later, Freeman, Odell, and Schlumprecht [15] showed that if X is any Banach space having separable dual, there exists a \mathcal{L}_∞ Banach space Z_X containing an isomorphic copy of X so that Z_X^* is isomorphic to ℓ_1 . Thus we see that not only does the question from [9] have a negative answer, but that the failure is quantitatively essentially as strong as possible. By this, we mean that for $1 < p < \infty$, $\mathcal{SM}_p(Z_p) = \infty$ while $\mathcal{SM}_q(Z_p^*) = \mathcal{SM}_q(\ell_1) = 1$ by the Schur property. Moreover, if $1 < p < 2$, $I_p(Z_p) = \infty$ while $I_q(Z_p^*) = \omega$, the smallest possible value. However, since ℓ_q is finitely representable in ℓ_1 for $1 \leq q \leq 2$, if $2 \leq p < \infty$, $I_p(Z_p) = \infty$ while $I_q(Z_p^*) = \omega^2$. Moreover, for $\xi < \omega_1$, with T being the Tsirelson space $T(\theta, \xi)$, to be defined later, $I_1(Z_T) \geq I_1(T) = \omega^\xi$, while $I_\infty(Z_T^*) = \omega$. Thus we can find separable Banach spaces with arbitrarily large, countable I_1 index, while the duals of these spaces have the smallest possible I_∞ index. Thus the presence of large ℓ_p structures in a given space does not imply the presence of a large ℓ_q structure in the dual.

5.2. Dualization for $p = \infty$. The obvious exception to the above argument is $p = \infty$. It is tempting to believe that the ℓ_1 structures in the dual of a Banach space must be at least as large as the c_0 indices in the space itself. For the indices involving an FDD, the result is clearly true. This is because if E is an FDD for X with projection constant b in X , then for any $x \in c_{00}(E)$, we can choose $x^* \in bB_{X^*} \cap c_{00}(E^*)$ with $\text{ran}_{E^*}(x^*) \subset \text{ran}_E(x)$ so that $\|x\| = x^*(x)$. In this way, we can construct block trees in $B_{[E^*]}$ the branches of which are normed by branches of $T_\infty(X, E, K)$ to verify that the order of $T_1([E^*], E^*, bK)$ must be at least as large as the order of $T_\infty(X, E, K)$. This gives the general idea for dualization of each index, which is to witness membership of sequences in $T_1(X^*, K)$ by norming them by an ∞ -absolutely convex combination of a member of $T_\infty(X, K)$ acting biorthogonally or almost biorthogonally on it. However, unlike the block case, it is not obvious how to choose the functionals. Of course, if $(x, x_1, \dots, x_n) \in T_\infty(X, K)$, we can take Hahn-Banach extensions $(x^*, x_1^*, \dots, x_n^*)$ of the biorthogonal functionals of this sequence. But say $(x, y_1, \dots, y_m) \in T_\infty(X, K)$. We can again obtain a sequence of Hahn-Banach extensions $(y^*, y_1^*, \dots, y_m^*)$ of the biorthogonals to this sequence, but the functionals x^* and y^* need not be the same. Thus we must work harder to choose the vectors acting biorthogonally on branches of $T_\infty(X, K)$ in a way that is consistent for vectors occurring in multiple members of $T_\infty(X, K)$. The argument also works for preduals of X with a slight modification using Helly's theorem [17]. Recall that Helly's theorem implies that if X is a Banach space, $F \subset X^*$

is finite, and $x^{**} \in X^{**}$, then for any $\varepsilon > 0$ there exists $x \in X$ with $\|x\| < \|x^{**}\| + \varepsilon$ so that for each $x^* \in F$, $x^{**}(x^*) = x^*(x)$.

Proposition 5.1. *Let X be a Banach space. Let $F \subset X^*$ be finite, $T = d^\alpha(T_\infty(X, K)(t_0))$ for some $\alpha \in \mathbf{Ord}$ and $t_0 \in T_\infty(X, K)$. For $0 < \xi \in \mathbf{Ord}$ and $n \in \mathbb{N}$, if $o(T) > \omega^\xi n$, there exist a B -tree \mathcal{T} on $[1, \omega^\xi n)$ with $o(\mathcal{T}) = \omega^\xi n$ and functions $f : \mathcal{T} \rightarrow T$, $\phi : \mathcal{T} \rightarrow X$, and $\phi^* : \mathcal{T} \rightarrow KB_{X^*}$ so that for each $t \in \mathcal{T}$,*

- (i) $f(t|_1) \wedge \dots \wedge f(t) \in T$,
- (ii) $\phi(t) \in \text{co}_\infty(f(t))$,
- (iii) for $s \in \mathcal{T}$ comparable to t , $\phi^*(s)(\phi(t)) = \delta_{st}$,
- (iv) for $x \in t_0$, $\phi^*(t)(x) = 0$,
- (v) for $x^* \in F$, $x^*(\phi(t)) = 0$.

Moreover, if X is a dual space, say $X = Y^*$, and if $C > K$, the conclusion holds with ϕ^* mapping \mathcal{T} into CB_Y .

Theorem 5.2. *Fix $\xi \in \mathbf{Ord}$ and a Banach space X .*

- (i) *If $X \in P_1^\vee(\xi, \infty)$, then the dual of X and any predual of X lies in $P_1^\vee(\xi, 1)$.*
- (ii) *Fix $i \in \{2, 3\}$. Suppose E is an FDD for X such that $(E, X) \in P_i^\vee(\xi, \infty)$. Then $(E^*, [E^*]) \in P_i^\vee(\xi, 1)$. If Y is a Banach space with FDD F so that $F^* = E$ and $[F^*] = X$, then $(F, Y) \in P_i^\vee(\xi, 1)$.*
- (iii) *If $X \in P_4^\vee(\xi, \infty)$, then the dual of X and any predual of X lies in $P_4^\vee(\xi, 1)$.*

Proof. If X^* contains ℓ_1 (respectively, if $X = Y^*$ and if Y contains ℓ_1), the statement is obvious. If c_0 embeds into X , then X^* has ℓ_1 as a quotient, and therefore as a subspace. Similarly, if c_0 embeds into $X = Y^*$, then ℓ_1 embeds into Y . Thus we can assume ℓ_1 does not embed into X^* (resp. Y), and consequently that c_0 does not embed into X .

(i) This follows immediately from Proposition 5.1 by taking $T = T_\infty(X, K)$ for some K such that $o(T_\infty(X, K)) > \omega^\xi$. We obtain a tree $(\phi^*(t))_{t \in \mathcal{T}}$ with $o(\mathcal{T}) = \omega^\xi$ and $(\phi(t))_{t \in \mathcal{T}}$ so that for each $t \in \mathcal{T}$, $(\phi(t|_i))_{i=1}^{|t|} \in T_\infty(X, K)$. Then $(K^{-1}\phi^*(t|_i))_{i=1}^{|t|} \in T_1(X, K)$, since any linear combination of this sequence can be appropriately normed by an ∞ -absolutely convex combination of $(\phi(t|_i))_{i=1}^{|t|}$. Thus $X^* \in P_1^\vee(\xi, 1)$. If $X = Y^*$, showing $Y \in P_1^\vee(\xi, 1)$ is similar.

(ii) and (iii) are trivial.

(iv) If X contains a c_0^ξ spreading model, it contains a seminormalized basic c_0^ξ spreading model (x_n) . The biorthogonal functionals to this basis form an ℓ_1^ξ spreading model in $[x_n]^*$. Taking Hahn-Banach extensions of these biorthogonals and scaling appropriately to make the sequence lie in B_{X^*} gives an ℓ_1^ξ spreading model in X^* .

In the case that $X = Y^*$, let $(x_n^*) \subset X^*$ be a bounded sequence acting biorthogonally on the K - c_0^ξ spreading model $(x_n) \subset X$. Let $C = \sup_n \|x_n^*\|$. By Helly's theorem, we can choose for each $n \in \mathbb{N}$ some $y_n \in (C + \varepsilon)B_Y$ so that for each $1 \leq m \leq n$, $x_m(y_n) = x_n^*(x_m) = \delta_{mn}$. Fix $\varepsilon \in (0, 1)$ and choose $(\varepsilon_n) \subset (0, 1)$ so that $\sum \varepsilon_n < \varepsilon$. Since (x_n) is weakly null, we can

pass to a subsequence of (x_n) and the corresponding subsequence of (y_n) and assume that for each $1 \leq n < m$, $|x_m(y_n)| < \varepsilon_m$. Then for $E \in \mathcal{S}_\xi$,

$$\begin{aligned} \left\| \sum_{n \in E} a_n y_n \right\| &\geq \left(\sum_{m \in E} \operatorname{sgn}(a_m) x_m \right) \left(\sum_{n \in E} a_n y_n \right) \\ &\geq \sum_{n \in E} |a_n| x_n(y_n) - \sum_{n \in E} \sum_{n > m \in E} |a_n| |x_m(y_n)| - \sum_{n \in E} \sum_{n < m \in E} |a_n| |x_m(y_n)| \\ &= \sum_{n \in E} |a_n| - \sum_{n \in E} \sum_{n < m \in E} |a_n| |x_m(y_n)| \\ &\geq \sum_{n \in E} |a_n| - \sum_{n \in E} |a_n| \sum_m \varepsilon_m > (1 - \varepsilon) \sum_{n \in E} |a_n|. \end{aligned}$$

Thus $((C + \varepsilon)^{-1} y_n)$ is an ℓ_1^ξ spreading model in Y . □

We note that we can record the following quantitative versions of the previous result. For any $X \in \mathbf{Ban}$ with FDD E in the appropriate cases, $\xi \in \mathbf{Ord}$, and $K \geq 1$,

- (i) $I_\infty(X, K) > \omega^\xi$ implies $I_1(X^*, K) > \omega^\xi$,
- (ii) $\mathcal{SM}_\infty(X, K) > \xi$ implies $\mathcal{SM}_1(X^*, 2K + \varepsilon) > \xi$,
- (iii) if E has projection constant b in X , $I_\infty(X, E, K) > \xi$ implies $I_1([E^*], E^*, bK) > \xi$,
- (iv) $I_\infty^a(X, E, K) > \xi$ implies $I_1^a([E^*], E^*, bK) > \xi$.

Moreover, the analogous statements for preduals hold for (ii), (iii), and (iv), and the analogous statement for preduals holds for (i) if we replace K by $K + \varepsilon$. We remark that the $2K + \varepsilon$ estimate in (ii) comes from the fact that we can guarantee the c_0^ξ spreading model (x_n) is basic with basis constant $(1 + \varepsilon)$, and $K^{-1} \leq \|x_n\|$. In this case, the norms of the biorthogonal functionals are bounded by $(2 + 2\varepsilon)K$ and satisfy a 1- ℓ_1 lower estimate.

Proof of Proposition 5.1. Suppose $o(T) > \omega$. Fix $m > |F|$. For each $k \in \mathbb{N}$, we can choose $(x_{ik})_{i=1}^{mk} \in T$. Since $m > |F|$, for each $1 \leq j \leq k$, we can choose

$$y_{jk} \in \operatorname{co}_\infty(x_{ik} : (j-1)m < i \leq jm) \cap \bigcap_{f \in F} \ker(f).$$

Write $t_0 = (u_1, \dots, u_l)$, and note that $(u_1, \dots, u_l, y_{1k}, \dots, y_{kk}) \in T_\infty(X, K)$. This means if f_{1k}, \dots, f_{kk} are defined on $[u_i, y_{jk} : 1 \leq i \leq l, 1 \leq j \leq k]$ by $f_{jk}(y_{ik}) = \delta_{ij}$ and $f_{jk}(u_i) = 0$ for each i , $\|f_{jk}\| \leq K$. Let $x_{jk}^* \in KB_{X^*}$ be a Hahn-Banach extension of f_{jk} . Recall that

$$\mathcal{T}_\omega = \{(k, k-1, \dots, j) : k \in \mathbb{N}, 1 \leq j \leq k\}.$$

define

$$\begin{aligned} f((k, k-1, \dots, j)) &= (x_{ik} : (j-1)m < i \leq jm), \\ \phi((k, \dots, j)) &= y_{jk}, \end{aligned}$$

and

$$\phi^*((k, \dots, j)) = x_{jk}^*.$$

This gives the $\xi = 1, n = 1$ case. In the case that $X = Y^*$, the only modification we need is to replace x_{jk}^* with some member $v_{jk} \in (K + \varepsilon)B_Y$ so that v_{jk} and x_{jk}^* agree on the vectors u_1, \dots, u_l and y_{1k}, \dots, y_{kk} , which we can do by Helly's theorem.

Next, suppose we have the result for trees of order exceeding $\omega^\xi k$, $1 \leq k \leq n$. Suppose $T = d^\alpha(T_\infty(X, K)(t_0))$ is such that $o(T) > \omega^\xi(n+1)$, and let $T_0 = d^{\omega^\xi}(T) = d^{\alpha+\omega^\xi}(T_\infty(X, K)(t_0))$. Note that $o(T_0) > \omega^\xi n$. Let \mathcal{T} be a B -tree on $[1, \omega^\xi n)$ with order $\omega^\xi n$ and let $f : \mathcal{T} \rightarrow T_0$, $\phi : \mathcal{T} \rightarrow X$, $\phi^* : \mathcal{T} \rightarrow KB_{X^*}$ satisfy the conclusions. For each $t \in \text{MAX}(\mathcal{T})$, note that

$$s_t := f(t|_1) \wedge \dots \wedge f(t) \in T_0 = d^{\omega^\xi}(T) = d^{\alpha+\omega^\xi}(T_\infty(X, K)(t_0)).$$

This means that $T_t := d^\alpha(T_\infty(X, K)(t_0 \wedge s_t))$ has $o(T_t) > \omega^\xi$. Apply the inductive hypothesis to obtain a B -tree \mathcal{T}_t on $[\omega^\xi n + 1, \omega^\xi(n+1))$ with $o(\mathcal{T}_t) = \omega^\xi$ and maps $f_t : \mathcal{T}_t \rightarrow T_t$, $\phi_t : \mathcal{T}_t \rightarrow X$, and $\phi_t^* : \mathcal{T}_t \rightarrow KB_{X^*}$ to satisfy the conclusions with F replaced by $F \cup \{\phi^*(t|_i) : 1 \leq i \leq |t|\}$ and t_0 replaced by $t_0 \wedge s_t$. Note that since for a fixed ζ , $\beta \mapsto \zeta + \beta$ is order preserving, we can assume the tree \mathcal{T}_t is on $[\omega^\xi n + 1, \omega^\xi(n+1))$ rather than $[1, \omega^\xi)$. We now let

$$\overline{\mathcal{T}} = \mathcal{T} \cup \{t \wedge s : t \in \text{MAX}(\mathcal{T}), s \in \mathcal{T}_t\}.$$

Note that $\overline{\mathcal{T}}$ is a B -tree on $[1, \omega^\xi(n+1))$ with $o(\overline{\mathcal{T}}) = \omega^\xi(n+1)$. We extend $f : \mathcal{T} \rightarrow T$, $\phi : \mathcal{T} \rightarrow X$, $\phi^* : \mathcal{T} \rightarrow KB_{X^*}$ to functions on $\overline{\mathcal{T}}$ by letting

$$f(t \wedge s) = f_t(s), \quad \phi(t \wedge s) = \phi_t(s), \quad \phi^*(t \wedge s) = \phi_t^*(s).$$

One easily checks that the conclusions are satisfied with these definitions.

Next, for each $n \in \mathbb{N}$, suppose the result holds for $\omega^\xi n$. Then if $o(T) > \omega^{\xi+1}$, $o(T) > \omega^\xi n$ for all $n \in \mathbb{N}$. Let $s_0 = 0$ and let $s_n = s_{n-1} + n$. Then for each $n \in \mathbb{N}$, there exists a B -tree $\mathcal{T}_{(n)}$ on $[1, \omega^\xi n)$ of order $\omega^\xi n$ and maps f_n, ϕ_n, ϕ_n^* satisfying the conclusions. Since $[1, \omega^\xi n)$ is order isomorphic to $[\omega^\xi s_{n-1} + 1, \omega^\xi s_n)$, we can assume $\mathcal{T}_{(n)}$ is a tree on $[\omega^\xi s_{n-1} + 1, \omega^\xi s_n)$. Then we let \mathcal{T} be the totally incomparable union of the $\mathcal{T}_{(n)}$. This is a B -tree on $[1, \omega^{\xi+1})$ with order $\omega^{\xi+1}$. We define f, ϕ, ϕ^* on \mathcal{T} by letting the restriction of each to $\mathcal{T}_{(n)}$ be equal to f_n, ϕ_n, ϕ_n^* , respectively.

Last, suppose the result holds for each $\zeta < \xi$, ξ a limit ordinal. Then for each $\zeta < \xi$, there exists a B -tree $\mathcal{T}_{(\zeta)}$ on $[1, \omega^{\zeta+1})$ having order $\omega^{\zeta+1}$ and maps $f_\zeta, \phi_\zeta, \phi_\zeta^*$ satisfying the conclusions. Since $[1, \omega^{\zeta+1})$ is order isomorphic to $[\omega^\zeta + 1, \omega^{\zeta+1})$, we can assume $\mathcal{T}_{(\zeta)}$ is actually a tree on $[\omega^\zeta + 1, \omega^{\zeta+1})$. We let \mathcal{T} be the totally incomparable union of $\mathcal{T}_{(\zeta)}$, $\zeta < \xi$, and let f, ϕ, ϕ^* be defined by letting the restriction to $\mathcal{T}_{(\zeta)}$ be $f_\zeta, \phi_\zeta, \phi_\zeta^*$, respectively. \square

6. CONSTANT REDUCTION FOR ℓ_1 AND c_0

It is clear that among normalized Schauder bases, the bases of ℓ_1 and c_0 play special roles. Every normalized Schauder basis is trivially 1-dominated by the ℓ_1 basis. Therefore if we wish to verify that a sequence $(x_i)_{i=1}^n \subset B_X$ is closely equivalent to the ℓ_1^n basis, we need only verify that this sequence has tight lower ℓ_1 estimates. A famous argument of James

[18] went as follows: If $(x_i)_{i=1}^{n^2} \subset B_X$ C^2 -dominates the $\ell_1^{n^2}$ basis, then there exists a block $(y_i)_{i=1}^n \subset B_X$ of $(x_i)_{i=1}^{n^2}$ which C -dominates the ℓ_1^n basis. Either there exists $1 \leq j \leq n$ so that for each $x \in \text{co}_1(x_i : (j-1)n < i \leq jn)$, $\|x\| \geq 1/C$, in which case we take $y_i = x_{(j-1)n+i}$. Otherwise we can take $z_j \in \text{co}_1(x_i : (j-1)n < i \leq jn)$ with $\|z_j\| < 1/C$. But $(z_j)_{j=1}^n$ also C^2 -dominates the ℓ_1^n basis, since it is a 1-absolutely convex block of a sequence which does, and so $(y_j)_{j=1}^n = (Cz_j)_{j=1}^n \subset B_X$ C -dominates the ℓ_1^n basis. Iterating this argument implies that if ℓ_1 is crudely finitely representable in X , then ℓ_1 is finitely representable in X . To restate in our language, $P_1^\vee(1, 1) = P_1^\wedge(1, 1)$.

In [19], a transfinite version of the dichotomy above was formulated to prove an extension of this result. Much of this section is contained either implicitly or explicitly in [19], while much of it is also new. Many results here, while closely related to those of [19], are more precise quantifications, extensions to uncountable ordinals and non-separable spaces, or have simplified proofs using our coloring lemmas.

We note that whereas the ℓ_1 upper estimates come automatically for sequences in B_X , c_0 lower estimates do not follow automatically for sequences (x_i) with $\|x_i\| \geq 1$. For this, we need another argument of James, also from [18]. If $\varepsilon \in [0, 1)$ and if $(x_i)_{i=1}^n$ is a sequence of vectors with $\|x_i\| \geq 1$ which is $(1 + \varepsilon)$ -dominated by the ℓ_∞^n basis, then $(x_i)_{i=1}^n$ $(1 - \varepsilon)^{-1}$ -dominates the ℓ_∞^n basis. To restate in our language, if $(x_i)_{i=1}^n \in W(X, 1 + \varepsilon)$ for $\varepsilon \in [0, 1)$, $((1 + \varepsilon)^{-1}x_i)_{i=1}^n \in T_\infty(X, (1 + \varepsilon)(1 - \varepsilon)^{-1})$. To see this, assume $(a_i)_{i=1}^n \in S_{\ell_\infty^n}$ and $a_j = 1$ for some $1 \leq j \leq n$. Let $w = \sum_{i=1}^n a_i x_i$ and $w' = a_j x_j - \sum_{i \neq j}^n a_i x_i$, then

$$2 \leq 2\|a_j x_j\| = \|w + w'\| \leq \|w\| + \|w'\| \leq \|w\| + 1 + \varepsilon.$$

This is the reason we content ourselves to study the trees $W(\cdot, \cdot)$ in place of $T_\infty(\cdot, \cdot)$.

We say an ordinal index I depending on $K \geq 1$ is *subadditive* provided if for all $C, K \geq 1$, $I(CK) \leq I(C) + I(K)$, and *submultiplicative* if $I(CK) \leq I(C)I(K)$. Of course, we automatically deduce that in this case we can reverse the order of addition or multiplication.

6.1. Submultiplicativity and consequences.

Theorem 6.1. *If X is any Banach space (with FDD E in the appropriate cases), each of the following indices is submultiplicative:*

- (i) $I_1(X, \cdot)$,
- (ii) $I_1(X, E, \cdot)$,
- (iii) $I_1^a(X, E, \cdot)$,
- (iv) $J(X, \cdot)$,
- (v) $J(X, E, \cdot)$,
- (vi) $J^a(X, E, \cdot)$.

Moreover, $\mathcal{SM}_1(X, \cdot)$ and $\mathcal{SJ}(X, \cdot)$ are subadditive.

Proof. (i) If $I_1(X, C) = \infty$ or $I_1(X, K) = \infty$, there is nothing to prove. Let $\zeta = I_1(X, C)$ and $\xi = I_1(X, K)$. Assume $I_1(X, CK) > \zeta\xi$. Then we can choose $(x_t)_{t \in \mathcal{T}_{\zeta\xi}}$ so that $(x_{t|_i})_{i=1}^{|t|} \in$

$T_1(X, CK)$ for all $t \in \mathcal{T}_{\zeta\xi}$. Define $f : C(\mathcal{T}_{\zeta\xi}) \rightarrow \{0, 1\}$ by $f(c) = 1$ provided

$$\min\{\|x\| : x \in \text{co}_1(x_t : t \in c)\} \geq 1/C,$$

and $f(c) = 0$ otherwise. If $i : \mathcal{T}_{\zeta} \rightarrow \mathcal{T}_{\zeta\xi}$ is order preserving so that $f \circ i(c) = 1$ for all $c \in C(\mathcal{T}_{\zeta})$, then $(x_{i(t)})_{t \in \mathcal{T}_{\zeta}}$ witnesses the fact that $I_1(X, C) > \zeta$, a contradiction. Therefore by Lemma 3.1, there exists an order preserving $j : \mathcal{T}_{\xi} \rightarrow C(\mathcal{T}_{\zeta\xi})$ so that $f \circ j \equiv 0$. For each $t \in \mathcal{T}_{\xi}$, we choose $u_t \in \text{co}_1(x_s : s \in j(t))$ with $\|u_t\| < 1/C$. Then since j is order preserving, $(u_{t|_i})_{i=1}^{|t|}$ is a 1-absolutely convex block of a member of $T_1(X, CK)$, and is therefore also a member of $T_1(X, CK)$. Then $(Cu_t)_{t \in \mathcal{T}_{\xi}}$ witnesses the fact that $I_1(X, K) > \xi$, another contradiction.

(ii) The proof is the same, except each $(x_{t|_i})_{i=1}^{|t|}$, and therefore each $(u_{t|_i})_{i=1}^{|t|}$, is a block sequence with respect to some FDD.

(iii) The proof is essentially the same, with the B -trees $\mathcal{T}_{\xi}, \mathcal{T}_{\zeta}, \mathcal{T}_{\zeta\xi}$ replaced by regular families $\mathcal{F}, \mathcal{G}, \mathcal{F}[\mathcal{G}]$ with $\iota(\mathcal{F}) = \xi$ and $\iota(\mathcal{G}) = \zeta$. We remark that in this case, in either of the alternatives above, the sequences of immediate successors do have minima of supports tending to infinity, as required. This is because Lemma 3.3 involves embeddings rather than order preserving maps.

(iv)-(vi) are essentially the same. We consider the function $f(c) = 1$ if

$$\max\{\|x\| : x \in \text{co}_{\infty}(x_t : t \in c)\} \leq C,$$

and 0 otherwise.

The remark about \mathcal{SM}_1 and \mathcal{SJ} is even easier. If (x_n) is a CK - $\ell_1^{\xi+\zeta}$ spreading model, we can pass to a subsequence which is a CK - $\ell_1^{\mathcal{S}_{\zeta}[\mathcal{S}_{\xi}]}$ spreading model. Either there exists $N \in \mathbb{N}$ so that for all $N \leq E \in \mathcal{S}_{\xi}$, $(x_n)_{n \in E} \in T_1(X, C)$, in which case $(x_n)_{n > N}$ is a C - \mathcal{S}_{ξ} spreading model, or there exist $E_1 < E_2 < \dots$, $E_i \in \mathcal{S}_{\xi}$, and scalars (a_i) so that for all $n \in \mathbb{N}$, $\sum_{i \in E_n} |a_i| = 1$ and $\|\sum_{i \in E_n} a_i x_i\| < 1/C$. Then let $y_n = \sum_{i \in E_n} a_i x_i$, so (Cy_n) is a K - ℓ_1^{ζ} spreading model. The proof for \mathcal{SJ} is similar. □

This yields the following: If ξ is any ordinal, and if $\zeta < \omega^{\omega^{\xi}}$, then $\zeta^n < \omega^{\omega^{\xi}}$ for every $n \in \mathbb{N}$. If $I_1(X, K) > \omega^{\omega^{\xi}}$ for some $K \geq 1$, and if $C > 1$, we can fix $n \in \mathbb{N}$ so that $K^{1/n} < C$. Then if $I_1(X, C) = \zeta < \omega^{\omega^{\xi}}$,

$$\omega^{\omega^{\xi}} < I_1(X, K) \leq I_1(X, C^n) \leq I_1(X, C)^n = \zeta^n < \omega^{\omega^{\xi}},$$

a contradiction. So $I_1(X, C) \geq \omega^{\omega^{\xi}}$ and, since $I_1(X, C)$ is a successor, $I_1(X, C) > \omega^{\omega^{\xi}}$. Thus if $X \in P_1^{\vee}(\omega^{\xi}, 1)$, $X \in P_1^{\wedge}(\omega^{\xi}, 1)$. We can perform similar arguments for $P_2^{\vee}(\omega^{\xi}, 1)$ and $P_3^{\vee}(\omega^{\xi}, 1)$.

Similarly, if $I_{\infty}(X, K) > \omega^{\omega^{\xi}}$, $J(X, K) > \omega^{\omega^{\xi}}$. As in the previous paragraph, we deduce that for any $\varepsilon \in (0, 1)$, $J(X, 1 + \varepsilon) > \omega^{\omega^{\xi}}$, which means $I_{\infty}(X, (1 + \varepsilon)(1 - \varepsilon)^{-1}) > \omega^{\omega^{\xi}}$. Thus we deduce $P_i^{\vee}(\omega^{\xi}, \infty) = P_i^{\wedge}(\omega^{\xi}, \infty)$ for $i = 1, 2, 3$.

If ξ is a limit ordinal, building a tree of order ξ is simply a matter of building a tree of order ζ for every $\zeta < \xi$ and taking a totally incomparable union. But for spreading models, this is not true. One must perform the blocking argument in a way which results in a sequence, not a tree. It was shown in [6] that this can be done to obtain the spreading model analogue of the results above. We sketch a proof, since it is an easy application of Lemma 3.5.

Theorem 6.2. *For any $0 \leq \xi \leq \omega_1$, $P_4^\vee(\omega^\xi, 1) = P_4^\wedge(\omega^\xi, 1)$ and $P_4^\vee(\omega^\xi, \infty) = P_4^\wedge(\omega^\xi, \infty)$.*

Proof. For $\xi = \omega_1$, the result is simply a restatement of the result of James. Suppose (x_n) is a $K\text{-}\ell_1^{\omega^\xi}$ spreading model. It is sufficient to show that some block of this sequence is a $K^{1/2}\text{-}\ell_1^{\omega^\xi}$ spreading model. We define $f : \widehat{\mathcal{S}}_{\omega^\xi} \rightarrow \{0, 1\}$ by letting $f(E) = 1$ if

$$\min\{\|x\| : x \in \text{co}_1(x_n : n \in E)\} \geq K^{-1/2},$$

and $f(E) = 0$ otherwise. Either there exists $M \in [\mathbb{N}]$ so that for each $E \in \widehat{\mathcal{S}}_{\omega^\xi}$, $f(M(E)) = 1$, in which case (x_{m_n}) is a $K^{1/2}\text{-}\ell_1^{\omega^\xi}$ spreading model, or there exist $E_1 < E_2 < \dots$ so that $f(E_i) = 0$ and so that for each $E \in \mathcal{S}_{\omega^\xi}$, $\cup_{i \in E} E_i \in \mathcal{S}_{\omega^\xi}$. For each $n \in \mathbb{N}$, choose $y_n \in \text{co}_1(x_i : i \in E_n)$ with $\|y_n\| < K^{-1/2}$. Then $(K^{1/2}y_n)$ is a $K^{1/2}\text{-}\ell_1^{\omega^\xi}$ spreading model.

For the c_0 case, we mimic the argument above with $c_0^{\omega^\xi}$ special sequences in place of $\ell_1^{\omega^\xi}$ spreading models. As before, a $(1 + \varepsilon)\text{-}c_0^{\omega^\xi}$ special sequence yields a $(1 + \varepsilon)(1 - \varepsilon)^{-1}\text{-}c_0^{\omega^\xi}$ spreading model. □

Corollary 6.3. *For $\xi \in \mathbf{Ord}$, $p \in \{1, \infty\}$, and $i \in \{1, 2, 3, 4\}$, $P_i^\vee(\omega^\xi, p) = P_i^\wedge(\omega^\xi, p)$.*

We can now define for $p \in \{1, \infty\}$ and $i \in \{1, 2, 3, 4\}$ $\phi_{p,i} : \mathbf{Ord} \rightarrow \mathbf{Ord}$ by

$$\phi_{p,i}(\xi) = \min\{\zeta \in \mathbf{Ord} : P_i^\vee(\zeta, p) \subset P_i^\wedge(\xi, p)\}.$$

Corollary 6.3 implies that this is well-defined, and

$$\phi_{p,i}(\xi) \leq \min\{\omega^\zeta : \omega^\zeta \geq \xi\}.$$

We next aim to discuss the sharpness of this estimate for certain values of i, p .

6.2. Schreier and Tsirelson spaces, the repeated averages hierarchy. For convenience in this section, we treat members of $[\mathbb{N}]^{<\omega}$ as projections on ℓ_∞ by letting Ex be the pointwise product of x and the sequence $(1_E(n))_n$. If $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is any regular family containing all singletons, the formula

$$\|x\|_{\mathcal{F}} = \sup_{i \in E} \|Ex\|_{\ell_1}$$

defines a norm making the canonical c_{00} basis normalized and 1-unconditional. The completion of c_{00} with respect to this norm is denoted $X_{\mathcal{F}}$. If $\mathcal{F} = \mathcal{S}_\xi$, we write X_ξ in place of $X_{\mathcal{S}_\xi}$ and $\|\cdot\|_\xi$ in place of $\|\cdot\|_{\mathcal{S}_\xi}$. This space is called the *Schreier space of order ξ* . This space was considered for $\xi = 1$ by Schreier [33], for finite values of ξ in [3], and for $\xi < \omega_1$ in [1].

Additionally, we have Banach spaces of the type of Tsirelson. Fix $\theta \in (0, 1)$. Define the sequence $(|\cdot|_n)_{n \geq 0}$ on c_{00} by letting $|\cdot|_0 = \|\cdot\|_{c_0}$ and

$$|x|_{n+1} = |x|_n \vee \sup \left\{ \theta \sum_{i=1}^s |E_i x|_n : (E_i)_{i=1}^s \text{ is } \mathcal{F}\text{-admissible} \right\}.$$

Since $|\cdot|_n \leq \|\cdot\|_{\ell_1}$, we note that $\lim_n |x|_n$ defines a norm on c_{00} making the canonical c_{00} basis normalized and 1-unconditional. We let $T(\theta, \mathcal{F})$ denote the completion of c_{00} under this norm. We note that the norm $\|\cdot\|_{T(\theta, \mathcal{F})}$ satisfies the implicit formula

$$\|x\|_{T(\theta, \mathcal{F})} = \|x\|_{c_0} \vee \sup \left\{ \theta \sum_{i=1}^s \|E_i x\|_{T(\theta, \mathcal{F})} : (E_i)_{i=1}^s \text{ is } \mathcal{F}\text{-admissible} \right\}.$$

For $\theta = 1/2$ and $\mathcal{F} = \mathcal{S}_1$, this is the Figiel-Johnson Tsirelson space [14], which is the dual of Tsirelson's original space [34]. As usual, we write $T(\theta, \xi)$ in place of $T(\theta, \mathcal{S}_\xi)$.

We next recall the repeated averages hierarchy, introduced in [4]. This will be the final tool we will need in order to discuss the sharpness of the results above. Given $L \in [\mathbb{N}]$ and $0 \leq \xi < \omega_1$, we will define a convex blocking $(x_n^{L, \xi})$ of the canonical c_{00} basis so that

- (i) $\text{supp}(x_n^{L, \xi}) \in \mathcal{S}_\xi$, and
- (ii) $\cup_n \text{supp}(x_n^{L, \xi}) = L$.

These sequences will allow us precise quantification of the complexity of blockings we will require to measure $I_1(X_\xi, K)$ and $I_1(T(\theta, \omega^\xi), K)$.

We let $L = (l_n)$ and $x_n^{L, 0} = e_{l_n}$. If $(x_n^{L, \xi})$ has been defined, let $s_0 = 0$ and $L_0 = L$. Recursively choose s_n, p_n, L_n so that for all $n \in \mathbb{N}$,

- (i) $p_n = \min L_{n-1}$,
- (ii) $s_n = p_n + s_{n-1}$,
- (iii) $L_n = L_{n-1} \setminus \cup_{i=s_{n-1}+1}^{s_n} \text{supp}(x_i^{L, \xi})$.

Then let $x_n^{L, \xi+1} = p_n^{-1} \sum_{i=s_{n-1}+1}^{s_n} x_i^{L, \xi}$.

Suppose that $\xi < \omega_1$ is a limit ordinal and that for every $M \in [\mathbb{N}]$ and every $\zeta < \xi$, $x_n^{M, \zeta}$ has been defined. Let $\xi_n \uparrow \xi$ be such that $\mathcal{S}_\xi = \{E : \exists n \leq E \in \mathcal{S}_{\xi_n+1}\}$. Let $L_0 = L$ and recursively define p_n, L_n so that for each $n \in \mathbb{N}$,

- (i) $p_n = \min L_{n-1}$,
- (ii) $L_n = L_{n-1} \setminus \text{supp}(x_1^{L_{n-1}, \xi_{p_n}+1})$.

Let $x_n^{L, \xi} = x_1^{L_{n-1}, \xi_{p_n}+1}$.

For $(m_n) = M \in [\mathbb{N}]$, we let $T_M : \ell_\infty \rightarrow \ell_\infty$ be the operator defined by $T e_n = e_{m_n}$. We observe that for $x \in c_{00}$ and $\xi < \omega_1$, $\|T_M x\|_\xi$ may be much larger than $\|x\|_\xi$, and that T_M does not necessarily map X_ξ into X_ξ . Given $M \in [\mathbb{N}]$ and $\varepsilon > 0$, we say $L \in [\mathbb{N}]$ is (M, ε) *fast growing* provided $m_{l_n}/l_{n+1} < \varepsilon/(1+2\varepsilon)$ for all $n \in \mathbb{N}$. Note that any infinite subset of an (M, ε) fast growing set is also (M, ε) fast growing, and any set $N \in [\mathbb{N}]$ admits $L \in [N]$ which is (M, ε) fast growing.

Lemma 6.4. *If L is (M, ε) fast growing, then for any $k \in \mathbb{N}$,*

$$\left\| T_M \sum_{n=1}^k x_n^{L, \xi} \right\|_{\xi} \leq 1 + \varepsilon.$$

Proof. For M, ε fixed, we will prove by induction on ξ that for any (M, ε) fast growing $L \in [\mathbb{N}]$ and any $E \in \mathcal{S}_{\xi}$,

$$\left\| ET_M \sum_{n=1}^{\infty} x_n^{L, \xi} \right\|_{\ell_1} \leq 1 + \varepsilon.$$

For $\xi = 0$, since $\|\cdot\|_0 = \|\cdot\|_{c_0}$, this case is trivial.

Write $x_n^{L, \xi+1} = p_n^{-1} \sum_{i=s_{n-1}+1}^{s_n} x_i^{L, \xi}$, where $p_n = \min \text{supp}(x_n^{L, \xi+1})$. Choose $E \in \mathcal{S}_{\xi+1}$ and write $E = \cup_{i=1}^s E_i$ with $(E_i)_{i=1}^s$ \mathcal{S}_1 -admissible. If $ET_M x_n^{L, \xi+1} = 0$ for all $n \in \mathbb{N}$, there is nothing to prove. So assume not, and let $r = \min(n : ET_M x_n^{L, \xi+1} \neq 0)$. Fix $n \in \mathbb{N}$ and choose $l_{k_0} < \dots < l_{k_n}$ so that $m_{l_{k_0}} \in E \cap T_M x_r^{L, \xi+1}$ and $l_{k_i} = p_{r+i}$ for $1 \leq i \leq n$. Note that $s \leq \min E \leq m_{l_{k_0}}$, so

$$\frac{s}{p_{r+n}} \leq \frac{m_{l_{k_0}}}{l_{k_n}} \leq \prod_{i=1}^n \frac{m_{l_{k_{i-1}}}}{l_{k_i}} \leq (\varepsilon/(1+2\varepsilon))^n.$$

Then

$$\begin{aligned} \left\| ET_M \sum x_n^{L, \xi+1} \right\|_{\ell_1} &\leq 1 + \sum_{n=r+1}^{\infty} p_n^{-1} \sum_{j=1}^s \left\| E_j T_M \sum_{j=s_{n-1}+1}^{s_n} x_j^{L, \xi} \right\|_{\ell_1} \\ &\leq 1 + \sum_{n=r+1}^{\infty} p_n^{-1} \sum_{j=1}^s (1 + \varepsilon) = 1 + (1 + \varepsilon) \sum_{n=1}^{\infty} \frac{s}{p_{r+n}} \\ &\leq 1 + (1 + \varepsilon) \sum_{n=1}^{\infty} (\varepsilon/(1+2\varepsilon))^n = 1 + \varepsilon. \end{aligned}$$

Suppose the result holds for every (M, ε) fast growing set and every ordinal strictly less than the limit ordinal ξ . Recall that for some sequence $\xi_n \uparrow \xi$, $\mathcal{S}_{\xi} = \{E : \exists n \leq E \in \mathcal{S}_{\xi_n+1}\}$, and that for each $n \in \mathbb{N}$, $\mathcal{S}_{\xi_n+1} \subset \mathcal{S}_{\xi_{n+1}}$. Let $x_n^{L, \xi} = x_1^{L_n, \xi_{p_n}+1}$ for some $L_n \in [L]$ and $p_n = \min \text{supp}(x_n^{L, \xi})$. Note that each L_n is (M, ε) fast growing. Fix $E \in \mathcal{S}_{\xi}$ and let $r = \min(n : ET_M x_n^{L, \xi} \neq 0)$. As in the previous case, we deduce that $1/p_{r+n} < (\varepsilon/(1+2\varepsilon))^n$ for each $n \in \mathbb{N}$. Moreover, if $m = m_{l_{k_0}} \in E \cap \text{supp}(T_M x_r^{L, \xi})$ and if $l_{k_n} = p_{r+n}$, we deduce that $E \in \mathcal{S}_{\xi_{m+1}}$ and that $m < p_{r+n}$. The second assertion can be seen by noting that $m/l_{k_n} < \varepsilon/(1+2\varepsilon) < 1$. This means that $E \in \mathcal{S}_{\xi_{m+1}} \subset \mathcal{S}_{\xi_{p_n}}$ for each $n > r$. Then recalling

that $x_1^{L_n, \xi_{p_n}+1} = p_n^{-1} \sum_{j=1}^{p_n} x_j^{L_n, \xi_{p_n}}$, we deduce

$$\begin{aligned} \|ET_M \sum x_n^{L, \xi}\|_{\ell_1} &\leq 1 + \sum_{n=r+1}^{\infty} \|ET_M x_1^{L_n, \xi_{p_n}+1}\|_{\ell_1} \\ &= 1 + \sum_{n=r+1}^{\infty} p_n^{-1} \|ET_M \sum_{j=1}^{p_n} x_j^{L_n, \xi_{p_n}}\|_{\ell_1} \\ &\leq 1 + (1 + \varepsilon) \sum_{n=1}^{\infty} 1/p_{n+r} \leq 1 + \varepsilon. \end{aligned}$$

□

We remark that even if $M = \mathbb{N}$, the fast growing condition is required to obtain the $(1+\varepsilon)$ -estimate. To see this, fix $n \in \mathbb{N}$ and an interval I with $|I| = n$ and minimum $m > n+1$. Let $J = [m+n, 2(m+n))$. Then with $x = (n+1)^{-1}1_{(n+1) \wedge I}$, $y = |J|^{-1}1_J$, and $E = [m, 2m) \in \mathcal{S}_1$,

$$\|x + y\|_1 \geq \|E(x + y)\|_{\ell_1} = \frac{n}{n+1} + \frac{m-n}{m+n},$$

which can be made arbitrarily close to 2 by auspicious choices of m and n .

6.3. Schreier spaces. Assume $0 < \xi < \omega_1$. It is clear that if (x_i) is a bimonotone basic sequence which is a K - ℓ_1^ξ spreading model, then (x_i) is also a Kn - $\ell_1^{(\mathcal{S}_\xi)^n}$ spreading model. Since any space containing a K - ℓ_1^ξ contains a sequence equivalent to ℓ_1 or a K - ℓ_1^ξ spreading model which is weakly null, we can in either case find a K - ℓ_1^ξ spreading model which is basic with basis constant almost 1, and therefore has projection constant not more than $2+\varepsilon$. Then any space containing a K - ℓ_1^ξ spreading model contains a $(2+\varepsilon)n$ - K - $\ell_1^{(\mathcal{S}_\xi)^n}$ spreading model. Similarly, if E is bimonotone, the proof of Proposition 4.1 implies that if $I_1(X, E, K) > \omega^\xi$, $I_1(X, E, nK) > \omega^\xi n$. We will use the Schreier spaces to demonstrate that this lower estimate on the growth rate is optimal. We state this as follows.

Proposition 6.5. *Let $0 < \xi < \omega_1$ and let E denote the basis of the Schreier space X_ξ . Then E is a 1 - ℓ_1^ξ spreading model, but if $1 \leq K < n \in \mathbb{N}$, $I_1(X_\xi, E, K) < \omega^\xi n$. In particular, X_ξ contains a K - $\ell_1^{(\mathcal{S}_\xi)^n}$ spreading model if and only if $K \geq n$.*

Proof. It is clear that if (x_i) is a normalized block sequence in X_ξ , and if $m_i = \max \text{supp}(x_i)$, then (x_i) is 1-dominated by (e_{m_i}) . Moreover, if $m_i \leq k_i$ for each $i \in \mathbb{N}$, (e_{m_i}) is 1-dominated by (e_{k_i}) . For $K \geq 1$, we let

$$\mathcal{G}_K = \{(m_i)_{i=1}^n \in [\mathbb{N}]^{<\omega} : (e_{m_i})_{i=1}^n \in T_1(X_\xi, E, K)\}.$$

By the properties mentioned above, \mathcal{G}_K is spreading, and clearly it is hereditary. Moreover, we claim that for each ζ ,

$$\{(\max \text{supp}(x_i))_{i=1}^n : (x_i)_{i=1}^n \in d^\zeta(T_1(X_\xi, E, K))\} = d^\zeta(\mathcal{G}_K).$$

By definition of \mathcal{G}_K , $\mathcal{G}_K \subset \{(\max \text{supp}(x_i))_{i=1}^n : (x_i)_{i=1}^n \in T_1(X_\xi, E, K)\}$. But since $(x_i)_{i=1}^n \in T_1(X_\xi, E, K)$ is 1-dominated by $(e_{\max \text{supp}(x_i)})_{i=1}^n$, we have $\mathcal{G}_K \supset \{(\max \text{supp}(x_i))_{i=1}^n : (x_i)_{i=1}^n \in T_1(X_\xi, E, K)\}$. This establishes the $\zeta = 0$ case. The successor and limit ordinal cases of the claim are trivial. If \mathcal{G}_K is not compact, then there must exist some $M \in [\mathbb{N}]$ so that $[M]^{<\omega} \subset \mathcal{G}_K$. If \mathcal{G}_K is compact, \mathcal{G}_K is regular. In this case, if $\iota(\mathcal{G}_K) \geq \omega^\xi n$, there would exist some $M \in [\mathbb{N}]$ so that $(\mathcal{S}_\xi)_n(M) \subset \mathcal{G}_K$. In either case, there exists $M \in [\mathbb{N}]$ so that $(\mathcal{S}_\xi)_n(M) \subset \mathcal{G}_K$. Fix $\varepsilon > 0$ so that $(1 + \varepsilon)K < n$ and choose $L \in [\mathbb{N}]$ so that L is (M, ε) fast growing. Then $E = \cup_{i=1}^n \text{supp}(x_i^{L, \xi}) \in (\mathcal{S}_\xi)_n$, and $\text{supp}(T_M \sum_{i=1}^n x_i^{L, \xi}) = M(E) \in \mathcal{G}_K$. But this means $\|T_M \sum_{i=1}^n x_i^{L, \xi}\|_\xi \geq n/K > 1 + \varepsilon$, while Lemma 6.4 tells us that $\|T_M \sum_{i=1}^n x_i^{L, \xi}\|_\xi \leq 1 + \varepsilon$. This proves that \mathcal{G}_K is compact with $\iota(\mathcal{G}_K) < \omega^\xi n$. Then as a consequence of our claim above, $o(T_1(X_\xi, E, K)) \leq \iota(\mathcal{G}_K) + 1 < \omega^\xi n$. \square

Note that the canonical basis E^* of X_ξ^* is a $1\text{-}c_0^\xi$ spreading model. But since the basis of X_ξ is bimonotone, $I_\infty(X_\xi^*, E^*, K) < \omega^\xi n$ for $K < n$ by the quantitative versions of Theorem 5.1. Thus we see the sharpness of this lower estimate on growth rate for the $p = \infty$ structures as well. Finally, the analogue of this statement for $1 < p < \infty$ and the p -convexification of X_ξ gives optimality of the lower bound on the growth rate of each of the indices for $1 < p < \infty$.

6.4. Tsirelson space. In this section, we aim to prove that for $i = 2, 3, 4$ and $p = 1, \infty$, $\phi_{p,i}(\xi) = \min\{\omega^\zeta : \omega^\zeta \geq \xi\}$ for countable ξ , yielding that Corollary 6.3 is sharp in some cases. We note that our methods rely on dealing with block sequences in a basis, so they do not fully elucidate the values $\phi_{1,1}(2)$ or $\phi_{\infty,1}(2)$. We will prove the following.

Theorem 6.6. *For $\xi < \omega_1$ and $\omega^\xi < \zeta < \omega^{\xi+1}$, $T(\theta, \omega^\xi) \in P_4^\vee(\zeta, 1)$, while $(E, T(\theta, \omega^\xi)) \notin P_2^\wedge(\omega^\xi + 1, 1)$. Here, E denotes the canonical basis of $T(\theta, \omega^\xi)$.*

We first recall that if E is an FDD for X and if $X \in P_1^\wedge(1 + \zeta, 1)$, then $(E, X) \in P_2^\wedge(\zeta, 1)$ and, if $\zeta \geq \omega$, if $X \in P_1^\wedge(\zeta, 1)$, then $(E, X) \in P_2^\wedge(\zeta, 1)$. Thus the argument will show that $T(\theta, 1) \notin P_1^\wedge(3, 1)$ and $T(\theta, \omega^\xi) \notin P_1^\wedge(\omega^\xi + 1, 1)$. This settles the values of $\phi_{1,i}(\xi)$ for all countable values of ξ if $i = 2, 3, 4$, and for all values of ξ except 2 when $i = 1$.

By dualization, this argument will settle the values of $\phi_{\infty,i}(\xi)$ for all countable ξ if $i = 2, 3, 4$ and for all countable values of ξ except 2 when $i = 1$. If $T(\theta, \omega^\xi)^* \in P_i^\wedge(\omega^\xi + 1, \infty)$ for $i = 2, 3, 4$, it would imply that $T(\theta, \omega^\xi) \in P_i^\wedge(\omega^\xi + 1, 1)$.

Note that the basis (e_n) of $T(\theta, \omega^\xi)$ is θ^{-1} -asymptotic $\ell_1^{\omega^\xi}$, and therefore θ^{-k} -asymptotic $\ell_1^{[\mathcal{S}_\xi]^k}$ for each $k \in \mathbb{N}$. If $\zeta < \omega^{\xi+1}$, there exists $k \in \mathbb{N}$ so that $\zeta < \omega^\xi k$. There exists $M \in [\mathbb{N}]$ so that $\mathcal{S}_\zeta(M) \subset [\mathcal{S}_\xi]^k$, so that (e_{m_n}) is a normalized θ^{-k} -asymptotic ℓ_1^ζ basis. Therefore it is a θ^{-k} - ℓ_1^ζ spreading model, and $T(\theta, \omega^\xi) \in P_4^\vee(\zeta, 1)$. A similar argument for the basis of $T(\theta, \omega^\xi)^*$, which is θ^{-1} -asymptotic $c_0^{\omega^\xi}$, gives that $T(\theta, \omega^\xi)^* \in P_4^\vee(\zeta, \infty)$ for each $\zeta < \omega^{\xi+1}$.

To complete the proof that $T(\theta, \omega^\xi)$, we will need to appeal to the following technical propositions. The first consists of now standard arguments.

Proposition 6.7. (i) Let $(x_i)_{i=1}^N$ be a block sequence in $B_{T(\theta, \mathcal{F})}$ and let $E_1 < \dots < E_n$ be any sets. Then

$$\sum_{j=1}^n \left\| E_j \sum_{i=1}^N x_i \right\|_{T(\theta, \mathcal{F})} \leq N + 2n.$$

(ii) Let (x_i) be a block sequence in $B_{T(\theta, \mathcal{F})}$ and let $m_i = \max \text{supp}(x_i)$. Then for any $(a_i) \in c_{00}$,

$$\left\| \sum a_i x_i \right\|_{T(\theta, \mathcal{F})} \leq \theta \sum |a_i| + \left\| \sum a_i e_{m_i} \right\|_{\mathcal{F}}.$$

Proof. (i) By 1-unconditionality, we can assume that for each j , E_j is an interval and that $\text{supp}(x_i) \subset \cup_{j=1}^n E_j$ for each i . For each $1 \leq j \leq n$, let $A_j = (i : E_j x_i \neq 0)$ and $B_j = (i : E_j x_i = x_i)$. Then $\sum_{j=1}^n |B_j| \leq N$ and $|A_j| \leq 2 + |B_j|$. Then

$$\sum_{j=1}^n \left\| E_j \sum_{i=1}^N x_i \right\|_{T(\theta, \omega^\xi)} \leq \sum_{j=1}^n \left\| \sum_{i \in A_j} x_i \right\|_{T(\theta, \omega^\xi)} \leq \sum_{j=1}^n |A_j| \leq 2n + \sum_{j=1}^n |B_j| \leq N + 2n.$$

Of course, we could replace $T(\theta, \mathcal{F})$ with any space having a bimonotone basis if we required that the sets E_j are intervals.

(ii) If $\left\| \sum a_i x_i \right\|_{T(\theta, \mathcal{F})} = \left\| \sum a_i x_i \right\|_{c_0} \leq \left\| \sum a_i e_{m_i} \right\|_{\mathcal{F}}$, we are done. If $\left\| \sum a_i x_i \right\|_{T(\theta, \mathcal{F})} > \left\| \sum a_i x_i \right\|_{c_0}$, there exists an \mathcal{F} -admissible sequence $(E_j)_{j=1}^s$ so that

$$\left\| \sum a_i x_i \right\|_{T(\theta, \mathcal{F})} = \theta \sum_{j=1}^s \left\| E_j \sum a_i x_i \right\|_{T(\theta, \mathcal{F})}.$$

Let

$$A = \{i \in \mathbb{N} : E_j x_i \neq 0 \text{ for exactly one value of } j\}$$

and

$$B = \{i \in \mathbb{N} : E_j x_i \neq 0 \text{ for at least two values of } j\}.$$

For $i \in A$, let j_i be the unique value of j so that $E_{j_i} x_i \neq 0$. For $i \in B$, let j_i be the maximum value of j so that $E_j x_i \neq 0$. Then $i \mapsto j_i$ defines an injection from B into $\{1, \dots, s\}$. Moreover, $\min E_{j_i} \leq m_i$ for $i \in B$. Thus $(m_i)_{i \in B}$ is a spread of $(\min E_{j_i})_{i \in B}$. We deduce that $(m_i)_{i \in B} \in \mathcal{F}$. Then

$$\begin{aligned} \left\| \sum a_i x_i \right\|_{T(\theta, \mathcal{F})} &= \theta \sum_{j=1}^s \left\| E_j \sum a_i x_i \right\|_{T(\theta, \mathcal{F})} \\ &= \theta \sum_{i \in A} |a_i| \|E_{j_i} x_i\|_{T(\theta, \mathcal{F})} + \sum_{i \in B} |a_i| \theta \sum_{j=1}^s \|E_j x_i\|_{T(\theta, \mathcal{F})} \\ &\leq \theta \sum |a_i| + \sum_{i \in B} |a_i| \leq \theta \sum |a_i| + \left\| \sum a_i e_{m_i} \right\|_{\mathcal{F}}. \end{aligned}$$

□

We are now ready to sketch the proof that $I_1(T(\theta, 1), E, K) < \omega^2$ for any $K < \theta^{-1}$. This proof uses rapidly increasing sequences of ℓ_1 averages, which we will see again in the next section. It is based on E. Odell's proof that $T(1/2, 1)$ is $(2 - \varepsilon)$ -distortable. If $I_1(T(\theta, 1), E, K) > \omega^2$, for any $N \in \mathbb{N}$, we could recursively choose $t_n = (x_i^n)_{i=1}^{l_n}$ so that for each $1 \leq n \leq N$, $t_1 \wedge \dots \wedge t_n \in d^{\omega(N-n)}(T_1(T(\theta, 1), E, K))$ and so that $\max \text{supp}(x_{l_n}^n)/l_{n+1} < \varepsilon$ for each $1 \leq n < N$. Let $x_n = l_n^{-1} \sum_{i=1}^{l_n} x_i^n$ and $x = N^{-1} \sum_{n=1}^N x_n$. Then $\|x\|_{T(\theta, 1)} \geq 1/K$, since it is a 1-absolutely convex combination of a member of $T_1(T(\theta, 1), E, K)$. Note that $\|x\|_{c_0} \leq 1/N$. If $\|x\|_{T(\theta, 1)} > \|x\|_{c_0}$, we can fix $(E_j)_{j=1}^s$ \mathcal{S}_1 -admissible so that

$$\|x\|_{T(\theta, 1)} = \theta \sum_{j=1}^s \|E_j x\|_{T(\theta, 1)}.$$

Let $r = \min(n : E_1 x_n \neq 0)$. Then $s \leq \min E_1 \leq \max \text{supp}(x_r)$. Therefore

$$\begin{aligned} \|x\|_{T(\theta, 1)} &\leq \theta N^{-1} \sum_{n=r}^N l_n^{-1} \sum_{j=1}^s \|E_j \sum_{i=1}^{l_n} x_i^n\|_{T(\theta, 1)} \\ &\leq N^{-1} \|x_r\|_{T(\theta, 1)} + \theta \sum_{n=r+1}^N l_n^{-1} (l_n + 2s) \\ &\leq N^{-1} + \theta(1 + 2\varepsilon)(N - 1)/N. \end{aligned}$$

Since we can do this for any $N \in \mathbb{N}$ and any $\varepsilon > 0$, this yields a contradiction when $K < \theta^{-1}$.

This settles the $\xi = 0$ case of Theorem 6.6. For $0 < \xi$, by the proof of Proposition 4.1, it is sufficient to compute the I_1^a index of $T(\theta, \omega^\xi)$. For this we will use the following, which is an easy consequence of Lemma 6.4

Proposition 6.8. *Suppose $m : \widehat{\mathcal{S}}_{\omega_1} \rightarrow \mathbb{N}$ is such that for $\emptyset \prec E \prec F$, $m(E) < m(F)$, and if $E < k < l$, $m(E \wedge k) < m(E \wedge l)$. Then for any $\rho > 0$ and $\xi < \omega_1$, there exist $F \in \mathcal{S}_{\xi+1}$ and scalars $(a_n)_{n=1}^{|F|}$ with $\sum_{n=1}^{|F|} |a_n| = 1$ and $\|\sum_{n=1}^{|F|} a_n e_{m(F|_n)}\|_\xi < \rho$.*

Proof. Fix $\varepsilon > 0$ and choose recursively $l_1 < l_2 < l_3 < \dots$ and $m_1 < m_2 < \dots$ so that for each $n \in \mathbb{N}$,

- (i) $l_1 > (1 + \varepsilon)\rho^{-1}$,
- (ii) $m((l_1, \dots, l_n)) \leq m_{l_n}$,
- (iii) $m_{l_n} < \varepsilon l_{n+1}/(1 + 2\varepsilon)$.

Then with $M = (m_n)$ and $L = (l_n)$, L is (M, ε) fast growing. Write $x_1^{L, \xi+1} = \sum_{i=1}^N a_i e_{l_i}$, and recall that $x_1^{L, \xi+1} = l_1^{-1} \sum_{i=1}^{l_1} x_i^{L, \xi}$. Then $F = (l_i)_{i=1}^N = \text{supp}(x_1^{L, \xi+1}) \in \mathcal{S}_{\xi+1}$. Moreover, since

$$\begin{aligned} m(F|_n) &= m((l_1, \dots, l_n)) \leq m_{l_n}, \\ \left\| \sum_{n=1}^N a_n e_{m(F|_n)} \right\|_\xi &\leq \left\| \sum_{n=1}^N a_n e_{m_{l_n}} \right\|_\xi = \|T_M x_1^{L, \xi+1}\|_\xi = l_1^{-1} \left\| T_M \sum_{n=1}^{l_1} x_n^{L, \xi} \right\|_\xi \leq \frac{1 + \varepsilon}{l_1} < \rho. \end{aligned}$$

□

Proof of Theorem 6.6. We have already treated the $\xi = 0$ case. If $(E, T(\theta, \omega^\xi)) \in P_2^\wedge(\omega^\xi + 1, 1)$, then $(E, T(\theta, \omega^\xi)) \in P_3^\wedge(\omega^\xi + 1, 1)$. This means that for $K < \theta^{-1}$, we can find an asymptotic block tree $(x_F)_{F \in \widehat{\mathcal{S}}_{\omega^\xi+1}}$ so that for each $F \in \widehat{\mathcal{S}}_{\omega^\xi+1}$, $(x_{F|i})_{i=1}^{|F|} \in T_1(T(\theta, \omega^\xi), E, K)$. Let $m(F) = \max \text{supp}(x_F)$. Extend $m : \widehat{\mathcal{S}}_{\omega^\xi+1} \rightarrow \mathbb{N}$ to any function $m : \widehat{\mathcal{S}}_{\omega_1} \rightarrow \mathbb{N}$ which still satisfies the hypotheses of Proposition 6.8. Choose $\rho > 0$ so that $\theta + \rho < K^{-1}$ and $F \in \mathcal{S}_{\omega^\xi+1}$, $(a_i)_{i=1}^{|F|}$ so that $\sum_{i=1}^{|F|} |a_i| = 1$ and $\|\sum_{i=1}^{|F|} a_i e_{m(F|i)}\|_{\omega^\xi} < \rho$. Then

$$\left\| \sum_{i=1}^N a_i x_{F|i} \right\|_{T(\theta, \omega^\xi)} \leq \theta + \left\| \sum_{i=1}^{|F|} a_i e_{m(F|i)} \right\|_{\omega^\xi} < \theta + \rho < K^{-1}.$$

This contradiction yields the result. \square

Note that we have actually shown that $I_1(T(\theta, \omega^\xi), K) < \omega^{\xi+1}$ when $K < \theta^{-1}$.

7. CONSTANT REDUCTION FOR REFLEXIVE ℓ_p

7.1. Positive results from Krivine's theorem. As was noted in [21] and [31], Krivine's theorem [20] implies the following result.

Theorem 7.1. *If E is any Schauder basis, there exists $1 \leq p \leq \infty$ so that for all $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists a normalized block sequence $(x_i)_{i=1}^n$ in E which is $(1 + \varepsilon)$ -equivalent to the unit vector basis of ℓ_p^n .*

It is clear that if the basis E is equivalent to the ℓ_q (resp. c_0) unit vector basis, then the p from Krivine's theorem can only be q (resp. ∞). These facts together are easily seen to imply the following quantitative version: For $K \geq 1$, $n \in \mathbb{N}$, $\varepsilon > 0$, and $1 \leq p \leq \infty$, there exists $N = N(n, K, \varepsilon, p)$ so that if $(e_i)_{i=1}^N$ is K -equivalent to the unit vector basis of ℓ_p^N , then there exists a normalized block $(x_i)_{i=1}^n$ of $(e_i)_{i=1}^N$ which is $(1 + \varepsilon)$ -equivalent to the ℓ_p^n basis. Indeed, if it were not so, we could find $K \geq 1$, $n \in \mathbb{N}$, and $\varepsilon > 0$ so that for every $N \in \mathbb{N}$, there exists $(e_i^N)_{i=1}^N$ K -equivalent to the unit vector basis of ℓ_p^N with no normalized block of length n being $(1 + \varepsilon)$ -equivalent to the ℓ_p^n basis. If \mathcal{U} is a free ultrafilter on \mathbb{N} , we define a norm on c_{00} by

$$\left\| \sum a_i e_i \right\|_{\mathcal{U}} = \lim_{N \in \mathcal{U}} \left\| \sum a_i e_i^N \right\|.$$

Then this norm is K -equivalent to the ℓ_p norm on c_{00} . By Theorem 7.1, there is a normalized block $(x_i)_{i=1}^n$ of the c_{00} basis which is $(1 + \varepsilon)^{1/2}$ -equivalent to the ℓ_p^n basis. But by standard arguments, this block is $(1 + \varepsilon)^{1/2}$ equivalent to a block of $(e_i^N)_{i=1}^N$ for some $N \in \mathbb{N}$. This contradiction gives the quantitative version.

This quantitative version of Krivine's theorem yields immediately that $P_1^\vee(1, p) = P_1^\wedge(1, p)$ and $P_2^\vee(1, p) = P_2^\wedge(1, p)$ for all $1 \leq p \leq \infty$. That is, crude finite representability of ℓ_p in a Banach space X is equivalent to finite representability of ℓ_p in X , and crude block finite representability is equivalent to block finite representability. This string of positive results continues with $P_3^\vee(1, p) = P_3^\wedge(1, p)$. To see this, suppose $I_p^a(X, E, K) > \omega$. Fix $\varepsilon > 0$,

$n \in \mathbb{N}$, and let $N = N(n, K, \varepsilon, p)$ be as above. Suppose $(x_F)_{F \in \widehat{\mathcal{A}}_N}$ is such that for each $k_1 < \dots < k_N$, $(x_{(k_1, \dots, k_i)})_{i=1}^N$ is a block in F which is K -equivalent to the ℓ_p^N basis, and that for each $F \in \mathcal{A}_N$, $k \leq \text{supp}_E(x_F \frown_k)$. Then define a norm on $(e_i)_{i=1}^N$ by

$$\left\| \sum_{i=1}^N a_i e_i \right\| = \lim_{k_1 \in \mathcal{U}} \dots \lim_{k_N \in \mathcal{U}} \left\| \sum_{i=1}^N a_i x_{(k_1, \dots, k_i)} \right\|.$$

By standard pruning arguments, we can pass to a pruned subtree and assume that every branch of the tree $(x_F)_{F \in \widehat{\mathcal{A}}_N}$ is $(1 + \varepsilon)$ -equivalent to $(e_i)_{i=1}^N$. Moreover, $(e_i)_{i=1}^N$ is K -equivalent to the ℓ_p^N basis, so there exist $0 = k_0 < \dots < k_n = N$ and scalars $(b_j)_{j=1}^N$ so that with $f_i = \sum_{j=k_{i-1}+1}^{k_i} b_j e_j$, $(f_i)_{i=1}^n$ is $(1 + \varepsilon)$ -equivalent to the unit vector basis of ℓ_p^n . We can then define a tree $(y_F)_{F \in \widehat{\mathcal{A}}_n}$ the blocks of which are blocks of branches of $(x_F)_{F \in \widehat{\mathcal{A}}_N}$, which we deduce to be $(1 + \varepsilon)^2$ -equivalent to the ℓ_p^n basis. To see this, define $G_{(i)} = (i + 1, \dots, i + k_1)$ and, if for each $1 \leq j \leq l < n$, $G_{(i_1, \dots, i_j)}$ has been defined so that $(G_{(i_1, \dots, i_j)})_{j=1}^l$ is successive and $|G_{(i_1, \dots, i_j)}| = k_j - k_{j-1}$, we let $m = \max G_{(i_1, \dots, i_l)}$ and for $i > i_l$, let

$$G_{(i_1, \dots, i_l, i)} = (m + i + 1, \dots, m + i + k_{l+1} - k_l).$$

Then for $F \in \widehat{\mathcal{A}}_n$, with $\cup_{i=1}^{|F|} G_{F|i} = (s_i)_{i=1}^{|F|}$, we let

$$y_F = \sum_{i=k_{|F|-1}+1}^{k_{|F|}} b_i x_{(s_1, \dots, s_i)}.$$

Then by our construction, $(y_{F|i})_{i=1}^n$ is $(1 + \varepsilon)$ -equivalent to $(f_i)_{i=1}^n$, and therefore $(1 + \varepsilon)^2$ -equivalent to the ℓ_p^n basis. With this, we deduce $P_3^\wedge(1, p) = P_3^\vee(1, p)$.

Modern proofs of Krivine's theorem use spreading models, and actually prove more than is stated in Theorem 7.1. Let us say that a block $(y_i)_{i=1}^n$ of (x_i) is a *constant distribution block* if there exists a sequence $(a_j)_{j=1}^N$ of scalars and natural numbers $m_1 < m_2 < \dots < m_{nN}$ so that for each $1 \leq i \leq n$, $y_i = \sum_{j=1}^N a_j x_{m_{(i-1)N+j}}$. We have a similar definition for infinite blocks (y_i) of (x_i) . We say that the Schauder basis (e_i) is *constant distribution block finitely representable in (x_i)* if for every $n \in \mathbb{N}$ and every $\varepsilon > 0$, there exists a constant distribution block $(y_i)_{i=1}^n$ of (x_i) which is $(1 + \varepsilon)$ -equivalent to $(e_i)_{i=1}^n$. We note that constant distribution finite block representability is transitive. Moreover, if \mathcal{U} is a free ultrafilter on \mathbb{N} , and if for each $N \in \mathbb{N}$, (y_i^N) is a normalized, constant distribution block of (x_i) , then the norm defined on c_{00} by

$$\left\| \sum a_i e_i \right\| = \lim_{N \in \mathcal{U}} \left\| \sum a_i y_i^N \right\|$$

defines a norm on c_{00} which is constant distribution block finitely representable in (x_i) . Moreover, it is clear that constant distribution blocks and spreading models of (x_i) are themselves constant distribution block finitely representable in (x_i) . Then modern proofs of Krivine's theorem for a normalized Schauder basis (x_i) such that neither ℓ_1 nor c_0 is block finitely representable in (x_i) involve first passing to a weakly Cauchy subsequence of (x_i) and then the seminormalized weakly null constant distribution block $(x_{2i} - x_{2i-1})$. Then one

passes to a spreading model and then uses the method above involving free ultrafilters several times to build further sequences, so that eventually one of these sequences is isometrically equivalent to the ℓ_p basis for some $1 < p < \infty$. At each step, each constructed basis was constant distribution block finitely representable in the previous basis, so that by transitivity we see that ℓ_p is constant distribution block finitely representable in X .

As before, we deduce the quantitative version: For $\varepsilon > 0$, $n \in \mathbb{N}$, $K \geq 1$, and $1 < p < \infty$, there exists $N = N(n, K, \varepsilon, p)$ so that if $(x_i)_{i=1}^N$ is any sequence K -equivalent to the unit vector basis of ℓ_p^N , there exists a constant distribution block $(y_i)_{i=1}^n$ of $(x_i)_{i=1}^N$ which is $(1 + \varepsilon)$ -equivalent to ℓ_p^n . Again, the proof is by contradiction. The only modification to the proof above we need is that when we construct the norm on c_{00} using a free ultrafilter, any finite, constant distribution block in (e_i) is $(1 + \varepsilon)$ -equivalent to a constant distribution block of $(x_i^N)_{i=1}^N$ for some N . We see the following application. Suppose that X is a Banach space and $K \geq 1$ is such that for some $1 < p < \infty$, X admits a K - $\ell_p^{A_N}$ spreading model. That is, for each $N \in \mathbb{N}$, there exists $(x_i) \subset X$ so that for every E with $|E| \leq N$, $(x_i)_{i \in E} \in T_p(X, K)$. Then for $n \in \mathbb{N}$ and $\varepsilon > 0$, we can choose $N = N(n, K, \varepsilon, p)$ and a K - $\ell_p^{A_N}$ spreading model (x_i) in X . By passing to a subsequence, we can assume that for every sequence of scalars $(a_i)_{i=1}^N$ and every $m_1 < \dots < m_N$ and $l_1 < \dots < l_N$,

$$\left\| \sum_{i=1}^N a_i x_{m_i} \right\| \leq (1 + \varepsilon) \left\| \sum_{i=1}^N a_i x_{l_i} \right\|.$$

Then there exists a constant distribution block of $(x_i)_{i=1}^N$, say $y'_i = \sum_{j=1}^m b_j x_{k_j^i}$, which is $(1 + \varepsilon)$ -equivalent to the ℓ_p^n basis, where the sets $E_i = (k_j^i)_{j=1}^m$ are successive subsets of $(1, \dots, N)$. But then if $y_i = \sum_{j=1}^m b_j x_{im+j}$, (y_i) is a constant distribution block of (x_i) so that for any E with $|E| \leq n$, (y_i) is $(1 + \varepsilon)^2$ -equivalent to $(y'_i)_{i=1}^n$. Therefore if we can find $\ell_p^{A_N}$ spreading models with a uniform constant, we can find them with constants arbitrarily close to 1. It follows from the fact that these block copies of ℓ_p in Krivine's theorem are found spreading models that if X is any infinite dimensional Banach space, either X admits $c_0^{A_N}$ spreading models uniformly (and therefore almost isometrically) or there exists $1 \leq p < \infty$ so that X admits $\ell_p^{A_N}$ spreading models uniformly (and therefore almost isometrically).

The previous paragraph emphasizes the difference between structures determined by trees and structures determined by sequences. Odell and Schlumprecht [25] demonstrated the existence of a Banach space X_{OS} admitting no c_0^1 or ℓ_p^1 spreading model. But X_{OS} admits uniform $\ell_1^{A_n}$ spreading models for all $n \in \mathbb{N}$. Moreover, it is easy to see that X_{OS}^* admits uniform $c_0^{A_n}$ spreading models and the p -convexification of X_{OS} admits uniform $\ell_p^{A_n}$ spreading models for all $n \in \mathbb{N}$.

7.2. Distortability and consequences. In [19], the following observation was made: For $1 < p < \infty$, there does not exist a function $\phi_p : \mathbf{Ord} \rightarrow \mathbf{Ord}$ so that $P_1^\vee(\phi_p(\xi), p) \subset P_1^\wedge(\xi, p)$ for all $\xi \in \mathbf{Ord}$. This is because the spaces ℓ_p , $1 < p < \infty$, are distortable. If X is K -isomorphic to ℓ_p , $I_p(X, K) = \infty$. If such a ϕ_p existed, this would mean $I_p(X, 1 + \varepsilon) = \infty$,

and X must admit a $(1 + \varepsilon)$ -isomorphic copy of ℓ_p . But this would imply that ℓ_p is not distortable. This leads to the following definitions. For $1 < p < \infty$ and $i \in \{1, 2, 3, 4\}$, let $M_i(p)$ denote the supremum of ordinals ξ so that there exists a function $\phi_{p,i} : [0, \xi] \rightarrow \mathbf{Ord}$ so that for each $1 \leq \zeta \leq \xi$, $P_i^\vee(\phi_{p,i}(\zeta), p) \subset P_i^\vee(\zeta, p)$. The observation above from [19] shows that $M_i(p)$ is well-defined. In [19], the problem of finding explicit estimates for $M_1(p)$ was raised. We will show the following.

Theorem 7.2. *Fix $1 < p < \infty$.*

- (i) $1 \leq M_1(p) \leq 2$.
- (ii) $1 = M_2(p) = M_3(p)$.
- (iii) $0 \leq M_4(p) \leq 1$.

We devote the remainder of this section to a quantification of the Odell-Schlumprecht distortion of ℓ_p . The argument is rather technical, so we give the outline first. Here, S will denote Schlumprecht space, which we define below. The equivalent norm on ℓ_p is defined by selecting a sequence $\varepsilon_k \downarrow 0$ and a sequence of sets $\mathcal{I}_k \subset c_{00} \cap B_S$ and $\mathcal{I}_k^* \subset c_{00} \cap B_{S^*}$ so that for each $k \in \mathbb{N}$ and each $x \in \mathcal{I}_k$, there exists $x^* \in \mathcal{I}_k^*$ so that $x^*(x) > 1 - \varepsilon_k$ and so that for $k, l \in \mathbb{N}$, $k \neq l$, $x \in \mathcal{I}_k$ and $x^* \in \mathcal{I}_l^*$, $|x^*(x)| < \varepsilon_{\min\{k, l\}}$. Then one uses these sets to construct a set $\mathcal{J}_k \subset S_{\ell_1}$ which almost intersects any block subspace of ℓ_1 , and then moves this set via the Mazur map into two sets $\mathcal{K}_k \subset S_{\ell_p}$ and $\mathcal{K}_k^* \subset B_{\ell_q}$, where q is the conjugate exponent to p . These sets will retain the property that any block subspace of ℓ_p almost intersects \mathcal{K}_k and so that if $x^* \in \mathcal{K}_k$ and $x \in \mathcal{K}_l$, with $k \neq l$, $|x^*(x)|$ will be small.

We let $\theta_k = 1/\log_2(k+1)$, and we let W be the minimal subset of c_{00} so that for each $n \in \mathbb{N}$ and each $E \in [\mathbb{N}]^{<\omega}$, $\pm e_n \in W$ and if $f_1 < \dots < f_n$, $f_i \in W$, then $\theta_n \sum_{i=1}^n E f_i \in W$. If $f_1 < \dots < f_n$, $f_i \in W$ and $f = \theta_n \sum_{i=1}^n f_i$, we say f has *weight* n . We can then define the Schlumprecht norm to be

$$\|x\|_S = \sup_{f \in W} |f(x)|.$$

From this, we can deduce that if $x_1 < \dots < x_n$, $\|\sum_{i=1}^n x_i\|_S \geq \theta_n \sum_{i=1}^n \|x_i\|_S$. From this, we deduce that ℓ_1 is block finitely representable in any block subspace of S , since 1 is the only possible value of p to satisfy the conclusion of Krivine's theorem in any block subspace of S . Then for $C > 1$ and $n \in \mathbb{N}$, we say $x \in S_S$ is a C - ℓ_1^n + average provided there exist $x_1 < \dots < x_n$ so that $\|x_i\| \leq C$ for each $1 \leq i \leq n$ and $x = n^{-1} \sum_{i=1}^n x_i$. By giving another quantitative version of Krivine's theorem, we deduce that for any $C > 1$ and any $n \in \mathbb{N}$, there exists $N = N(C, \varepsilon)$ so that for any block sequence of the basis of S of length N , the span of this block sequence contains a C - ℓ_1^n + average. For $\varepsilon > 0$ and $n \in \mathbb{N}$, we say a sequence $(x_i)_{i=1}^k$ is an RIS sequence of length k with constant $(1 + \varepsilon)$ if x_i is a $(1 + \varepsilon)$ - $\ell_1^{k_i}$ + average and k_i is much larger than $|\text{supp}(x_j)|$ for $1 \leq j < i \leq k$. Of course, precise quantifications must be made, for which we refer the reader to [24]. We say x is a $(1 + \varepsilon)$ RIS average of length k if x is the normalization of $\sum_{i=1}^k x_i$, where $(x_i)_{i=1}^k$ is an RIS sequence of length k with constant

$(1 + \varepsilon)$. The importance of such vectors is the following: If $n \in \mathbb{N}$ and $f_1 < \dots < f_n$, $f_i \in W$, and if $x \approx k^{-1} \sum_{i=1}^k x_i$ is a $(1 + \varepsilon)$ - ℓ_1^k average,

$$\theta_n \left(\sum_{j=1}^n f_j \right) \left(k^{-1} \sum_{i=1}^k x_i \right) \lesssim \theta_n (1 + \varepsilon) (1 + 2n/k).$$

This value is small if k is much larger than n . Thus long ℓ_1 -averages cannot take large values under functionals with small weights. But if n is much larger than $|\text{supp}(x)|$,

$$\theta_n \left(\sum_{j=1}^n f_j \right) \left(\sum_{i=1}^n x_i \right) \leq \theta_n |\text{supp}(x)|,$$

which is small. Thus we can deduce that functionals with weights which are too small or too large (depending on k and $|\text{supp}(x)|$, respectively) will give x a small value. Thus if we construct a sequence $(x_i)_{i=1}^k$ so that k_i is much larger than $|\text{supp}(x_j)|$ for each $1 \leq j < i \leq n$, then any functional in W must have a weight which is either too large or too small for all but one of the members of the sequence $(x_i)_{i=1}^k$. Moreover, since ℓ_1 is block finitely representable in every block subspace of S , for any $k \in \mathbb{N}$ and any $\varepsilon > 0$, we can find in any block subspace an RIS sequence of length k with constant $(1 + \varepsilon)$. This process naturally suggests that we require averages of averages to witness the distortion with this method. We will see below that we can find $(1 + \varepsilon)$ - ℓ_1^k -vectors in any sufficiently long normalized block in ℓ_p , and then we must average a sufficiently long sequence of these averages to obtain RIS sequences.

For any $\varepsilon_k \downarrow 0$, we can fix $s_k \in \mathbb{N}$, $s_k \uparrow \infty$, and $\delta_k > 0$ so that if $\mathcal{I}_k \subset S_S$ consists of all $(1 + \delta_k)$ RIS averages of RIS sequences of length s_k , and if $\mathcal{I}_k^* \subset B_{X^*}$ consists of vectors of the form $\theta_{s_k} \sum_{j=1}^{s_k} x_j^*$, where $(x_j)_{j=1}^{s_k}$ is a block sequence in B_{X^*} , then

- (i) for each $k \in \mathbb{N}$ and each $x \in \mathcal{I}_k$, there exists $x^* \in \mathcal{I}_k^*$ so that $x^*(x) \geq 1 - \varepsilon_k$,
- (ii) for each $j \neq k$, each $x \in \mathcal{I}_k$, and each $x^* \in \mathcal{I}_j^*$, $|x^*(x)| \leq \varepsilon_{\min\{j, k\}}$, and
- (iii) for each $k \in \mathbb{N}$ and any block subspace X of S , $X \cap \mathcal{I}_k \neq \emptyset$.

If we are interested in quantification, however, we can state the following quantified version of (iii) above: If $(x_t)_{t \in \mathcal{T}_{\omega^2}} \subset S_S$, then for any $k \in \mathbb{N}$, there exists $t \in \mathcal{T}_{\omega^2}$ and $x \in \mathcal{I}_k \cap \text{span}(x_{t_i} : 1 \leq i \leq |t|)$. We choose recursively t_1, \dots, t_k so that $t_1 \wedge \dots \wedge t_i \in d^{\omega(s_k - i)}(\mathcal{T}_{\omega^2})$ so that $|t_i|$ is sufficiently long to guarantee the existence of a $(1 + \varepsilon)$ - $\ell_1^{k_i}$ -average in $\text{span}(x_{t_i|j} : 1 \leq j \leq |t_i|)$, where k_i depends on the choices of $(1 + \varepsilon)$ - $\ell_1^{k_j}$ -averages for $1 \leq j < i$. In what follows, if $x = (a_n)$ and $y = (b_n)$ are scalar sequences, we will let $xy = (a_n b_n)$, $|x| = (|x_n|)$, and $x^r = (\text{sgn}(x_n) |x_n|^r)$ for $r > 0$. Recall the definition of the Mazur map, $M_p : S_{\ell_1} \rightarrow S_{\ell_p}$, $M_p x = x^{1/p}$, and that this is a uniform homeomorphism [29]. We define the sets $\mathcal{J}_k \subset S_{\ell_1}$, $\mathcal{K}_k \subset S_{\ell_p}$, and $\mathcal{K}_k^* \subset S_{\ell_q}$ by

$$\mathcal{J}_k = \left\{ \frac{u^* u}{\|u^* u\|_{\ell_1}} : u \in \mathcal{I}_k, u^* \in \mathcal{I}_k^*, \|u^* u\|_{\ell_1} \geq 1 - \varepsilon_k \right\},$$

$$\mathcal{K}_k = M_p(\mathcal{J}_k),$$

$$\mathcal{K}_k^* = M_q(\mathcal{J}_k).$$

We remark that since the basis of Schlumprecht space is 1-unconditional, the sets \mathcal{K}_k and \mathcal{K}_k^* are 1-unconditional. By this, we mean that $a \in \mathcal{K}_k$ if and only if $|a| \in \mathcal{K}_k$.

If $2 \leq p$, let $\varepsilon'_k = \varepsilon_k^{2/p}/(1 - \varepsilon_1)$, and if $1 < p < 2$, let $\varepsilon'_k = \varepsilon_k^{2/q}/(1 - \varepsilon_1)$. Note that if $x = M_p(s) \in \mathcal{K}_k = M_p(\mathcal{J}_k)$, then $x^* = M_q(s) = \mathcal{K}_k^* = M_q(\mathcal{J}_k)$, and $x^*(x) = 1$. Thus for each $k \in \mathbb{N}$ and $x \in \mathcal{K}_k$, there exists $x^* \in \mathcal{K}_k^*$ so that $x^*(x) = 1$. It was shown in [24] that if $k \neq j$ and if $x^* \in \mathcal{K}_k^*$, $x \in \mathcal{K}_j$, then $|x^*(x)| \leq \varepsilon'_{\min\{j,k\}}$. Thus we have the ℓ_p analogues of (i) and (ii) above. What remains is a quantified version of (iii). The facts in the next lemma are contained in [24], without the quantification of the supports of the blocking required.

Lemma 7.3. (i) For $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $N = N(\varepsilon, k)$ so that if $(y_i)_{i=1}^N \subset S_{\ell_1}$ is a block sequence, there exist $y \in \text{co}_1(y_i : 1 \leq i \leq N)$, a $(1 + \varepsilon)$ ℓ_1^n average $u \in S_S$, and $u^* \in B_{S^*}$ so that $\|y - u^*u\|_{\ell_1} < \varepsilon$ and $\text{supp}(u) \subset \text{supp}(y)$.
(ii) If $(y_t)_{t \in \mathcal{T}_{\omega^2}} \subset S_{\ell_1}$ is a block tree, $\varepsilon > 0$, and $j \in \mathbb{N}$, there exists $t \in \mathcal{T}_{\omega^2}$, $y \in \text{co}_1(y_{t|i} : 1 \leq i \leq |t|)$ and $y_0 \in \mathcal{J}_j$ so that $\|y - y_0\|_{\ell_1} < \varepsilon$.
(iii) If $(x_F)_{t \in \mathcal{T}_{\omega^2}} \subset S_{\ell_1}$ is a block tree, $\varepsilon > 0$, and $j \in \mathbb{N}$, there exists $t \in \mathcal{T}_{\omega^2}$, $x \in \text{co}_p(x_{t|i} : 1 \leq i \leq |t|)$ and $x_0 \in \mathcal{J}_j$ so that $\|x - x_0\|_{\ell_p} < \varepsilon$.

Proof. (i) We note that the statement here is stronger than the statement given in [24], but the statement here is implicit in the proof of Lemma 3.5 from [24].

(ii) As outlined above, we recursively choose $t_1 < \dots < t_{s_j}$, $t_1 \wedge \dots \wedge t_i \in d^{\omega(s_j-i)}(\mathcal{T}_{\omega^2})$, $y_i \in \text{co}_1(y_{t_i|k} : 1 \leq k \leq |t_i|)$, $u_i \in S_S$, and $u_i^* \in B_{S^*}$ according to (i) so that the normalization u of $\sum_{i=1}^{s_j} u_i$ is a member of \mathcal{I}_j , $u^* = (1 + \rho)^{-1} \theta_{s_j} \sum_{i=1}^{s_j} u_i^* \in \mathcal{I}_j^*$, where $\rho > 0$ is chosen small depending on ε , $\|u_i^* u_i - y_i\|_{\ell_1} < \varepsilon/2$, and $\|u^* u\|_{\ell_1} > 1 - \varepsilon_j$. Then $y_0 = \frac{u^* u}{\|u^* u\|_{\ell_1}} \in \mathcal{J}_j$ and, with $y = s_j^{-1} \sum_{i=1}^{s_j} y_i$,

$$\|y - y_0\|_{\ell_1} \leq \|y - u^* u\|_{\ell_1} + \|u^* u - y_0\|_{\ell_1} < 2(\varepsilon/2) = \varepsilon.$$

(iii) This follows from (ii) and the fact that the Mazur map $M_p : S_{\ell_1} \rightarrow S_{\ell_p}$ is a uniform homeomorphism which preserves supports and takes 1-absolutely convex combinations of block sequences to p -absolutely convex combinations of block sequences. \square

Lemma 7.4. Fix $k \in \mathbb{N}$ and let X_k be the completion of c_{00} under the norm

$$\|x\| = \max\{\varepsilon'_k \|x\|_{\ell_p}, \sup_{x^* \in \mathcal{K}_k^*} x^*(x)\}.$$

Then if E is the c_{00} basis, $I_p(X_k, E, K) < \omega^2$ for any $1 \leq K < (\varepsilon'_k)^{-1}$, and $I_p(X_k, E, (\varepsilon'_k)^{-1}) = \infty$.

Proof. Note that $\varepsilon'_k \|x\|_{\ell_p} \leq \|x\| \leq \|x\|_{\ell_p}$, which gives the last statement.

Since $I_p(X, E, K)$ is a successor, it is sufficient to show that if $I_p(X, E, K) > \omega^2$, $K \geq (\varepsilon'_k)^{-1}$. If $I_p(X, E, K) > \omega^2$, we can find a block tree $(g_t)_{t \in \mathcal{T}_{\omega^2}}$ so that for each $t \in \mathcal{T}_{\omega^2}$, $(g_{t|i})_{i=1}^{|t|} \in T_p(X, E, K)$.

Fix $\delta > 0$. We can apply Corollary 3.10 and assume there exists $\theta > 0$ so that for every $t \in \mathcal{T}_{\omega^2}$, $\theta \leq \|g_t\|_{\ell_p} \leq \theta(1 + \delta)$. Let $x_t = g_t / \|g_t\|_{\ell_p}$. Choose $t \in \mathcal{T}_{\omega^2}$, $x \in \text{co}_p(x_{t|_i} : 1 \leq i \leq |t|)$, and $x_0 \in \mathcal{K}_k$ so that $\|x - x_0\|_{\ell_p} < \delta$. Then as we noted earlier, $\|x_0\| = 1$ and therefore $\|x\| \geq 1 - \|x - x_0\| \geq 1 - \|x - x_0\|_{\ell_p} > 1 - \delta$. Let $(a_i)_{i=1}^{|t|} \in S_{\ell_p^{|t|}}$ be such that $x = \sum_{i=1}^{|t|} a_i x_{t|_i}$, and let $g = \sum_{i=1}^{|t|} a_i g_{t|_i}$. Then

$$\begin{aligned} 1 &\geq \|g\| = \left\| \sum_{i=1}^{|t|} a_i g_{t|_i} \right\| \\ &= \left\| \sum_{i=1}^{|t|} a_i \|g_{t|_i}\|_{\ell_p} x_{t|_i} \right\| \geq \theta \left\| \sum_{i=1}^{|t|} a_i x_{t|_i} \right\| \\ &= \theta \|x\| \geq \theta(1 - \delta), \end{aligned}$$

and $\theta \leq (1 - \delta)^{-1}$. Here we have used 1-unconditionality of the c_{00} basis with respect to $\|\cdot\|$.

Next, choose any $j > k$. Choose $s \in \mathcal{T}_{\omega^2}$, $x' \in \text{co}_p(x_{s|_i} : 1 \leq i \leq |s|)$, and $x'_0 \in \mathcal{K}_j$ so that $\|x' - x'_0\|_{\ell_p} < \delta$. Then $\|x'_0\| \leq \varepsilon'_k$, and $\|x'\| \leq \varepsilon'_k + \delta$. Choose $(b_i)_{i=1}^{|s|} \in S_{\ell_p^{|s|}}$ so that $x' = \sum_{i=1}^{|s|} b_i x_{s|_i}$ and let $g' = \sum_{i=1}^{|s|} b_i g_{s|_i}$. Then

$$\begin{aligned} K^{-1} &\leq \|g'\| = \left\| \sum_{i=1}^{|s|} b_i g_{s|_i} \right\| \leq \theta(1 + \delta) \left\| \sum_{i=1}^{|s|} b_i x_{s|_i} \right\| \\ &\leq \theta(1 + \delta)(\varepsilon'_k + \delta) \leq \frac{(1 + \delta)(\varepsilon'_k + \delta)}{1 - \delta}, \end{aligned}$$

where again we have used 1-unconditionality. Since $\delta > 0$ was arbitrary, we deduce that $K^{-1} \leq \varepsilon'_k$. □

Proof of Theorem 7.2. We have already shown that $M_1(p), M_2(p), M_3(p) \geq 1$. We have just shown that $M_2(p) = 1$, so that $M_4(p), M_3(p) \leq 1$. We have shown in the proof of Proposition 4.1 (v) that for any Banach space X with FDD E and any $K \geq 1$ and $\varepsilon > 0$, $I_p(X, K) \leq \omega I_p^a(X, E, K + \varepsilon)$, from which we deduce $M_1(p) \leq 1 + M_2(p) = 2$. □

We note here that this is the strongest possible result about the negative results concerning existence of uniform structures given the existence of crude structures. That is, for each $\xi < \omega^2$, there exists a constant K such that if X is any Banach space with $I_p(X) > \omega$, then $I_p(X, K) > \xi$. It follows from the proof of Proposition 4.1 (iv) where if $\xi < \omega n$, $n \in \mathbb{N}$, then we can take $K = 2n + \varepsilon$ for any $\varepsilon > 0$.

8. SUBSPACE AND QUOTIENT ESTIMATES

As we have seen already, there is a strong dichotomy between the ℓ_p and c_0 cases and the ℓ_p cases for $1 < p < \infty$ because we must only be concerned with one-sided estimates when

$p = 1$ or ∞ . We investigate here another instance when we deduce positive results due to only needing to verify a one-sided estimate: The case of passing to a quotient. If X is a Banach space and Y is a closed subspace, any sequence 1-dominated by the ℓ_p basis in X is sent by the quotient map $Q : X \rightarrow X/Y$ to a sequence also 1-dominated by the ℓ_p basis. Thus we have the following dichotomy: Given an ℓ_p structure in X , the image of this ℓ_p structure under the quotient map must also be an ℓ_p structure, or p -absolutely convex blocks of the branches must have small quotient norm. In the first case, we find an ℓ_p structure in X/Y , and in the second case, we find an ℓ_p structure which is only a small perturbation from being in Y . Assuming the perturbation is small enough, we will find an ℓ_p structure in Y . We note that again, ℓ_1 plays a special role for two reasons: The first is that ℓ_1 structures in X/Y imply ℓ_1 structures in X . The second is that small, uniform perturbations of sequences behaving like the ℓ_1 basis also behave like the ℓ_1 basis, but this is not true for the ℓ_p , $1 < p < \infty$, or c_0 bases. This is easily seen by considering $(\theta e_n + f_n) \subset \ell_1 \oplus \ell_p$, where $\theta > 0$ is small, (e_n) is the ℓ_1 basis, and (f_n) is the ℓ_p basis.

If X is a Banach space containing a copy of ℓ_p , and if Y is any closed subspace, then either Y or X/Y contains a copy of ℓ_p . To see this, suppose $(x_n) \subset X$ is equivalent to the ℓ_p basis. Let $Q : X \rightarrow X/Y$ denote the quotient map. We consider two cases. In the first case, there exist $N \in \mathbb{N}$ and $\varepsilon > 0$ so that for all $(a_i)_{i=N+1}^\infty \in c_{00} \cap S_{\ell_p}$, $\|\sum_{i=N+1}^\infty a_i Qx_i\| \geq \varepsilon$. In this case, $(x_i)_{i>N}$, $(Qx_i)_{i>N}$, and the ℓ_p basis are all equivalent. In the second case, for any $\varepsilon_n \downarrow 0$, we can find a p -absolutely convex block (u_n) of (x_n) so that for each $n \in \mathbb{N}$, $\|Qu_n\| < \varepsilon_n$. If we choose $(y_n) \subset Y$ so that $\|u_n - y_n\| < \varepsilon_n$, and if ε_n is chosen tending to 0 rapidly enough, depending on the constant of equivalence between (x_n) and the ℓ_p basis, (y_n) will be equivalent to the ℓ_p basis. More precisely, in the second case, if (x_n) is K -equivalent to the ℓ_p basis, then (u_n) is K -equivalent to the ℓ_p basis. For any $C > 0$, by choosing $\varepsilon_n \downarrow 0$ appropriately we can guarantee that the equivalence constant between (y_n) and the ℓ_p basis is at most C .

As a consequence of this, we deduce that the class of Banach spaces not containing ℓ_p is closed under taking direct sums. To restate, if $I_p(Y), I_p(Z) < \infty$, then $I_p(Y \oplus Z) < \infty$. Dodos has shown that this result is actually uniform on **SB** [12]. That is, there exists a function $\psi_p : [0, \omega_1)^2 \rightarrow [0, \omega_1)$ so that for $Y, Z \in \mathbf{SB}$, $I_p(Y \oplus Z) \leq \psi_p(I_p(Y), I_p(Z))$. This was shown using descriptive set theoretic techniques, and was actually shown for a larger class of spaces than the ℓ_p spaces. The methods used do not provide explicit estimates of the function ψ_p , and in [12], Dodos asks for some explicit estimate. We provide the first explicit estimate for this function below. Note that we do not require separability, nor do we assume countability of any ordinals.

Theorem 8.1. *Fix $1 \leq p \leq \infty$. For any Banach space X , any closed subspace Y , and $1 \leq K < C$,*

- (i) $I_p(X, K) \leq I_p(X/Y)I_p(Y, C)$.
- (ii) $I_1(X, K) \leq I_1(X/Y, 3K)I_1(Y, 3K)$.

- (iii) $\mathcal{SM}_p(X, K) \leq \mathcal{SM}_p(X/Y) + \mathcal{SM}_p(Y, C)$.
- (iv) $\mathcal{SM}_1(X, K) \leq \mathcal{SM}_1(X/Y, 3K) + \mathcal{SM}_1(Y, 3K)$.
- (v) $\mathcal{SM}_\infty(X) \leq \max\{\mathcal{SM}_\infty(X/Y), \mathcal{SM}_\infty(Y)\}$.

Corollary 8.2. *For any ordinal ξ , $I_1(\cdot) < \omega^{\omega^\xi}$ and $I_1(\cdot) \leq \omega^{\omega^\xi}$ are three-space properties on the class **Ban**.*

This is because if $\alpha, \beta < \omega^{\omega^\xi}$, $\alpha\beta < \omega^{\omega^\xi}$. This observation gives the following corollary as well.

Corollary 8.3. *Fix $1 \leq p \leq \infty$. If $X \in \mathbf{Ban}$ and Y is a closed subspace of X so that $I_p(X/Y), I_p(Y) < \omega^{\omega^\xi}$, then $I_p(X) < \omega^{\omega^\xi}$.*

In the case that Y is complemented, we can reverse the order of multiplication. In this case, we obtain the slightly stronger result that if $I_p(Y), I_p(Z) \leq \omega^{\omega^\xi}$ and at least one of these inequalities is strict, then $I_p(Y \oplus Z) \leq \omega^{\omega^\xi}$.

Of course, $I_p(\cdot) < \omega^{\omega^\xi}$ is not a three-space property for $1 < p$, since ℓ_p and c_0 are quotients of ℓ_1 . Thus for $1 < p$, $I_p(X/Y)$ cannot be controlled by $I_p(X)$.

As we will see below, the $\mathcal{SM}_1(X) = \sup_{K \geq 1} \mathcal{SM}_1(X, K)$ may be attained. This fact has been hinted at by the existence of Banach spaces admitting uniform $\ell_1^{A_n}$ spreading models but no ℓ_1^1 spreading model. Thus the estimate (iv) of Theorem 8.1 does not imply the following, which was proven in [6].

Theorem 8.4. *For $\xi < \omega_1$, admitting no $\ell_1^{\omega^\xi}$ spreading model is a three space property on the class of Banach spaces. That is, if $X \in \mathbf{Ban}$ and $Y \leq X$, $\mathcal{SM}_1(Y), \mathcal{SM}_1(X/Y) \leq \omega^\xi$ implies $\mathcal{SM}_1(X) \leq \omega^\xi$.*

This is a simple consequence of Lemma 3.5. If (x_n) is a K - $\ell_1^{\omega^\xi}$ spreading model in X , define the function f on $\widehat{\mathcal{S}}_{\omega^\xi}$ by $f(E) = 0$ if $\min\{\|Qx\| : x \in \text{co}_1(x_n : n \in E)\} \geq 1/3K$ and $f(E) = 1$ otherwise. If there exists $M \in [\mathbb{N}]$ so that $f|_{\widehat{\mathcal{S}}_{\omega^\xi}(M)} \equiv 1$, then the image (Qx_{m_n}) of the subsequence (x_{m_n}) is a $3K$ - $\ell_1^{\omega^\xi}$ spreading model. Otherwise there exist $E_1 < E_2 < \dots$ so that $f(E_i) = 0$ and for each $E \in \mathcal{S}_{\omega^\xi}$, $\cup_{n \in E} E_n \in \mathcal{S}_{\omega^\xi}$. For each $n \in \mathbb{N}$, pick $u_n \in \text{co}_1(x_i : i \in E_n)$ so that $\|Qu_n\| < 1/3K$ and $y_n \in Y$ so that $\|u_n - y_n\| < 1/3K$. Then (u_n) is a K - $\ell_1^{\omega^\xi}$ spreading model. This implies $(2^{-1}y_n)$ is a $3K$ - $\ell_1^{\omega^\xi}$ spreading model.

Proof of Theorem 8.1. (i) If $I_p(X, K) = \infty$, then X contains a K -isomorphic copy of ℓ_p . We have already treated this case above, so assume $I_p(X, K) < \infty$. If $I_p(X/Y) = \infty$ or $I_p(Y, C) = \infty$, there is nothing to prove. Therefore we can assume $\zeta = I_p(X/Y) < \infty$ and $\xi = I_p(Y, C) < \infty$. Assume $I_p(X, K) > \zeta\xi$ and choose $(x_t)_{t \in \mathcal{T}_{\zeta\xi}}$ so that for each $t \in \mathcal{T}_{\zeta\xi}$, $(x_{t|_i})_{i=1}^{|t|} \in T_p(X, K)$. Define $f : C(\mathcal{T}_{\zeta\xi}) \rightarrow \mathbb{R}$ by

$$f(c) = \inf\{\|Qx\| : x \in \text{co}_p(x_t : t \in c)\}.$$

By Lemma 3.1, either there exists an order preserving $i : \mathcal{T}_\zeta \rightarrow \mathcal{T}_{\zeta\xi}$ so that

$$\inf\{f \circ i(c) : c \in C(\mathcal{T}_\zeta)\} = \varepsilon > 0,$$

or there exists an order preserving $j : \mathcal{T}_\xi \rightarrow C(\mathcal{T}_{\zeta\xi})$ so that for each $t \in \mathcal{T}_\xi$, $f \circ j(t) < \varepsilon_{|t|}$, where $(\varepsilon_n) \subset (0, 1)$ is to be determined.

In the first case, $(Qx_{i(t)})_{t \in \mathcal{T}_\zeta}$ witnesses the fact that $I_p(X/Y) > \zeta$, a contradiction. In the second case, we can choose for each $t \in \mathcal{T}_\xi$ some $u_t \in \text{co}_p(x_s : s \in j(t))$ so that $\|Qu_t\| < \varepsilon_{|t|}$. Then since j is order preserving, $(u_{t|_i})_{i=1}^{|t|}$ is a p -absolutely convex block of a member of $T_p(X, K)$, and is itself a member of $T_p(X, K)$. For each $t \in \mathcal{T}_\xi$, we can choose $y_t \in Y$ so that $\|u_t - y_t\| < \varepsilon_{|t|}$. Then for each $t \in \mathcal{T}_\xi$ and $(a_i)_{i=1}^{|t|} \in S_{\ell_p^{|t|}}$,

$$\begin{aligned} \left\| \sum_{i=1}^{|t|} a_i y_{t|_i} \right\| &\leq \left\| \sum_{i=1}^{|t|} a_i u_{t|_i} \right\| + \sum_{i=1}^{|t|} |a_i| \|u_{t|_i} - y_{t|_i}\| \\ &\leq 1 + \sum_{i=1}^{|t|} \varepsilon_{t|_i} \leq 1 + \sum \varepsilon_n. \end{aligned}$$

Similarly, $\left\| \sum_{i=1}^{|t|} a_i y_{t|_i} \right\| \geq 1/K - \sum \varepsilon_n$. If $\varepsilon = \sum \varepsilon_n$ is chosen so that $(1+\varepsilon)(K^{-1}-\varepsilon)^{-1} < C$, we deduce that $((1+\varepsilon)^{-1}y_{t|_i})_{i=1}^{|t|} \in T_p(Y, C)$. This means $I_p(Y, C) > \xi$, another contradiction.

(ii) If X contains a K -isomorphic copy of ℓ_1 , then either X/Y or Y contains a $(1+\varepsilon)$ -isomorphic copy of ℓ_1 for all $\varepsilon > 0$. Assume $I_1(X, K) < \infty$, which implies that $\zeta = I_1(X/Y, 3K)$, $\xi = I_1(Y, 3K) < \infty$. The proof is similar to the previous case. If $I_1(X, K) > \zeta\xi$, choose $(x_t)_{t \in \mathcal{T}_{\zeta\xi}}$ so that $(x_{t|_i})_{i=1}^{|t|} \in T_1(X, K)$. Define a function $f : C(\mathcal{T}_{\zeta\xi}) \rightarrow \{0, 1\}$ by letting $f(c) = 1$ if

$$\inf\{\|Qx\| : x \in \text{co}_1(x_t : t \in c)\} \geq 1/3K,$$

and $f(c) = 0$ otherwise. Either there exists an order preserving $i : \mathcal{T}_\zeta \rightarrow \mathcal{T}_{\zeta\xi}$ so that $f \circ i(c) = 1$ for all $c \in C(\mathcal{T}_\zeta)$, in which case $I_1(X/K, 3K) > \zeta$, or there exists an order preserving $j : \mathcal{T}_\xi \rightarrow C(\mathcal{T}_{\zeta\xi})$ so that $f \circ j \equiv 0$. The first case immediately yields a contradiction. In the second case, we choose for each $t \in \mathcal{T}_\xi$ some $u_t \in \text{co}_1(x_s : s \in j(t))$ so that $\|Qu_t\| < 1/3K$ and some $y_t \in Y$ so that $\|u_t - y_t\| < 1/3K$. Then for any $t \in \mathcal{T}_\xi$ and $(a_i)_{i=1}^{|t|} \in S_{\ell_1^{|t|}}$,

$$\left\| \sum_{i=1}^{|t|} a_i y_{t|_i} \right\| \geq 1/K - \sum_{i=1}^{|t|} |a_i| \|u_{t|_i} - y_{t|_i}\| \geq 2/3K.$$

But $\|y_t\| \leq \|u_t\| + 1/3K < 2$, so $(2^{-1}y_t)_{t \in \mathcal{T}_\xi} \subset B_Y$ witnesses the fact that $I_1(Y, 3K) > \xi$, another contradiction.

(iii) If any of the three spaces contains ℓ_p , we finish as in (i). Assume each index does not exceed ω_1 . In this case, if either $\mathcal{SM}_p(X/Y) = \omega_1$ or $\mathcal{SM}_p(Y, C) = \omega_1$, we have the result. So assume $\zeta = \mathcal{SM}_p(X/Y) < \omega_1$ and $\xi = \mathcal{SM}_p(Y, C) < \omega_1$. If $\mathcal{SM}_p(X, K) > \zeta + \xi$, then there must exist some $(x_n) \subset X$ which is a K - $\ell_p^{S_\xi[S_\zeta]}$ (resp. $c_0^{S_\xi[S_\zeta]}$ if $p = \infty$) spreading model. Suppose that for some $N \in \mathbb{N}$ and some $\varepsilon > 0$ it is true that for every $E \in \mathcal{S}_\zeta \cap (N, \infty)^{<\omega}$ and

every $(a_n)_{n \in E} \in S_{\ell_p^{|E|}}$, $\varepsilon \leq \|\sum_{n \in E} a_n Qx_n\|$. Then $(Qx_n)_{n > N}$ is a ε^{-1} - $\ell_p^{\mathcal{S}_\zeta \cap (N, \infty)^{<\omega}}$ spreading model, a contradiction. Thus we can find $E_1 < E_2 < \dots$, $E_i \in \mathcal{S}_\zeta$, and $(a_j)_{j \in E_i} \in S_{\ell_p^{|E_i|}}$ so that with $u_i = \sum_{j \in E_i} a_j x_j$, $\|Qu_i\| < \varepsilon_i$, where $\varepsilon_i \downarrow 0$ rapidly. Then (u_i) is a K - ℓ_p^ξ spreading model and, if $y_i \in Y$ is such that $\|u_i - y_i\| < \varepsilon_i$ for all $i \in \mathbb{N}$, $((1 + \varepsilon)^{-1}y_i) \subset Y$ will be a C - ℓ_p^ξ spreading model for an appropriate choice of (ε_i) .

(iv) The proof is the obvious combination of the methods of (ii) and (iii).

(v) Again, we need only consider the case in which X does not contain c_0 . Fix $\xi < \omega_1$ and assume $(x_n) \subset X$ is a c_0^ξ spreading model. If $\xi = 0$, there is nothing to prove. Otherwise, (x_n) is weakly null. If there exists $N \in \mathbb{N}$ so that $(Qx_n)_{n > N}$ is bounded away from zero, then some subsequence is a c_0^ξ spreading model. This is because some subsequence is basic, so the lower c_0 estimates come automatically. The upper c_0 estimates come from comparison to $(x_n)_{n > N}$. Otherwise, there exists a subsequence of (x_n) , which we can assume is the whole sequence, so that $\|Qx_n\| \rightarrow 0$. Then some subsequence is an arbitrarily small perturbation of a sequence in Y . By taking the perturbations small enough, we guarantee the existence of a c_0^ξ spreading model in Y . \square

There is a natural exception to the remark above about uniform perturbations of ℓ_p sequences not necessarily exhibiting ℓ_p behavior. This exception is uniform perturbations of sequences with uniformly bounded length. If $I_p(X/Y), I_p(Y) \leq \omega$, the estimate above only yields $I_p(X) \leq \omega^2$. In this case one can perform the finite version of the arguments above to easily see that if $I_p(X) > \omega$, $\max\{I_p(X/Y), I_p(Y)\} > \omega$. This is because if $I_p(X) > \omega$ while $I_p(X/Y), I_p(Y)$, then for any $\varepsilon > 0$ and $N \in \mathbb{N}$, we can find $(x_i)_{i=1}^N \in T_p(X, 1 - \varepsilon)$. Fix $n \in \mathbb{N}$ arbitrary. With $m = I_p(X/Y, n/\varepsilon)$ and $N = mn$, we deduce that for each $1 \leq j \leq n$, there exists $(a_i)_{i=(j-1)m+1}^{jm} \in S_{\ell_p^m}$ so that $\|Q \sum_{i=(j-1)m+1}^{jm} a_i x_i\| < \varepsilon/n$. With $u_i = \sum_{i=(j-1)m+1}^{jm} a_i x_i$ and $y_i \in Y$ so that $\|u_i - y_i\| < \varepsilon/n$,

$$1 - 2\varepsilon \leq \left\| \sum_{i=1}^n b_i y_i \right\| \leq 1 + \varepsilon$$

for any $(b_i)_{i=1}^n \in S_{\ell_p^n}$. Since we can do this for any $n \in \mathbb{N}$, we deduce $I_p(Y) > \omega$.

We now deduce a few natural consequences of our coloring lemmas which help us estimate the I_p index of infinite direct sums. Recall that if U is a Banach space with 1-unconditional (not necessarily ordered) basis E , and if for each $e \in E$, X_e is a Banach space,

$$(\oplus X_e)_{e \in E} = \{(x_e)_{e \in E} : x_e \in X_e, \sum_{e \in E} \|x_e\| e \in U\}$$

is a Banach space with the norm $\|(x_e)\| = \|\sum_{e \in E} \|x_e\| e\|_U$. We denote this direct sum by X_E . Analogously to the case of an FDD, we let $\text{supp}_E((x_e)_{e \in E}) = \{e \in E : x_e \neq 0\}$ and we let $c_{00}(E)$ denote the vectors in X_E which have finite support. We note that such vectors are dense in X_E . For $S \subset E$ finite, we let $P_S : X_E \rightarrow X_E$ be the projection given by $P_S(x_e) = (1_S(e)x_e)$. We let $\pi_{e_0} : (\oplus X_e)_{e \in E} \rightarrow X_{e_0}$ be the coordinate projection into the

summand X_{e_0} defined by $\pi_{e_0}((x_e)_{e \in E}) = y_{e_0}$. Last, we let $\Pi : X_E \rightarrow U$ be the map defined by $\Pi(x) = \sum_{e \in E} \|\pi_e(y)\|e$. We observe that if (x_i) is a sequence of vectors with pairwise disjoint supports in X_E , then (x_i) is isometrically equivalent to the sequence (Πx_i) in U .

Proposition 8.5. *Fix $1 \leq p \leq \infty$. Suppose that U is a Banach space with 1-unconditional basis E and $\alpha \in \mathbf{Ord}$ is such that for each $e \in E$, X_e is a Banach space with $I_p(X_e) < \omega^{\omega^\alpha}$ (or $I_p(X_e) \leq \omega^{\omega^\alpha}$ in the case that $p = 1$). Then for any $1 \leq K < C$, $I_p(X_E, K) \leq \omega^{\omega^\alpha} I_p(U, C)$.*

Proof. (i) Let $\zeta = \omega^{\omega^\alpha}$. If $I_p(U, C) = \infty$, there is nothing to prove. So assume $\xi = I_p(U, C) < \infty$ and $I_p(Y, K) > \zeta \xi$. By replacing any K with any strictly larger number which is still less than C , we deduce there exists $(x_t)_{t \in \mathcal{T}_{\zeta \xi}}$ so that for each $t \in \mathcal{T}_{\zeta \xi}$, $(x_{t|_i})_{i=1}^{|t|} \in T_p(X, K) \cap c_{00}(E)^{<\omega}$. For $c \in C(\mathcal{T}_{\zeta \xi})$, let $S_c = \cup_{s \preceq \max c} \text{supp}_E(x_s)$, and note that this is a finite subset of E . Let $S_\emptyset = \emptyset$. For each $c \in C(\mathcal{T}_{\zeta \xi}) \cup \{\emptyset\}$, define $f_c : C(\mathcal{T}_{\zeta \xi}) \rightarrow \mathbb{R}$ by

$$f_c(c') = \min\{\|P_{S_c}x\| : x \in \text{co}_p(x_s : s \in c')\}.$$

Note that for any order preserving $i : \mathcal{T}_\zeta \rightarrow \mathcal{T}_{\zeta \xi}$ and for any $c \in C(\mathcal{T}_{\zeta \xi}) \cup \{\emptyset\}$,

$$\inf\{f_c \circ i(c') : c' \in C(\mathcal{T}_\zeta)\} = 0.$$

If this were not so, then with $0 < \varepsilon = \inf\{f_c \circ i(c') : c' \in C(\mathcal{T}_\zeta)\}$, we deduce $(P_{S_c}x_{i(s)})_{s \in \mathcal{T}_\zeta} \subset X_{S_c}$ witnesses the fact that $I_p(X_{S_c}, \varepsilon^{-1}) > \zeta$. But since X_{S_c} is a finite direct sum, this contradicts Theorem 8.1.

Fix (ε_n) rapidly decreasing to zero and fix an order preserving $j : \mathcal{MT}_\xi \rightarrow C(\mathcal{T}_{\zeta \xi}) \cup \{\emptyset\}$ so that for each $s \prec t \in \mathcal{T}_\xi$, $f_{j(s)}(j(t)) < \varepsilon_{|t|}$, which we can do by Lemma 3.2. Fix $t \in \mathcal{T}_\xi$ and let s be the immediate predecessor of t in \mathcal{MT}_ξ . Choose $y_t \in \text{co}_p(x_r : r \in j(t))$ so that $\|P_{S_{j(s)}}y_t\| < \varepsilon_{|t|}$, which we can do since $f_{j(s)}(j(t)) < \varepsilon_{|t|}$. Let $A_t = \text{supp}_E(y_t) \setminus S_{j(s)}$. Let $z_t = P_{A_t}y_t$ and note that $\|z_t - y_t\| = \|P_{S_{j(s)}}y_t\| < \varepsilon_{|t|}$. Thus for an appropriate choice of $\varepsilon_n \downarrow 0$ and an appropriate scaling of (z_t) , we deduce that $(z_t)_{t \in \mathcal{T}_\xi} \in T_p(X_E, C)$. But for any $s' \preceq s \prec t$, $\text{supp}_E(z_{s'}) \subset \text{supp}_E(y_{s'}) \subset S_{j(s)}$, so that $z_{s'}$ and z_t have disjoint supports. Thus for each $t \in \mathcal{T}_\xi$, $(\Pi z_{t|_i})_{i=1}^{|t|} \in T_p(U, C)$, since it is isometrically equivalent to $(z_{t|_i})_{i=1}^{|t|}$. This contradiction finishes the proof. \square

Proposition 8.6. *If U is a reflexive Banach space with 1-unconditional basis E and if $\alpha \leq \omega_1$ is such that for each $e \in E$, X_e is a Banach space with $\mathcal{SM}_p(X_e) < \omega^\alpha$ (or $\mathcal{SM}_p(X_e) \leq \omega^\alpha$ if $p = 1$), then for any $1 \leq K < C$,*

$$\mathcal{SM}_p(X_E, K) \leq \omega^\alpha \mathcal{SM}_p(U, C).$$

Proof. First, suppose that (x_n) K -dominates and is 1-dominated by the ℓ_p basis. By replacing K with a strictly larger number, we can assume that for each $n \in \mathbb{N}$, $S_n = \cup_{i=1}^n \text{supp}_E(x_i)$ is finite. Note that for any finite $S \subset E$, any $\varepsilon > 0$, and any $(m_n) = M \in [\mathbb{N}]$, there exists $F \in [\mathbb{N}]^{<\omega}$ and $(a_n)_{n \in F} \in S_{\ell_p^{|F|}}$ so that $\|P_S \sum_{n \in F} a_n x_{m_n}\| < \varepsilon$. If it were not so, then $(P_S x_n)_{n \in M}$ would be equivalent to the ℓ_p basis in the finite direct sum X_S . But since

no space X_e contains ℓ_p , the finite direct sum X_S cannot. Thus for any $\varepsilon_n \downarrow 0$, we can recursively choose a p -absolutely convex block $y_n = \sum_{i \in F_n} a_i x_i$ so that for each $n \in \mathbb{N}$, $\|P_{S_{\max F_{n-1}}} y_n\| < \varepsilon_n$. Then with $z_n = y_n - P_{S_{\max F_{n-1}}} y_n$, we obtain a small perturbation of (y_n) which is disjointly supported in X_E and $(K + \varepsilon)$ -equivalent to the ℓ_p basis. Then $(\Pi z_n) \subset U$ is $(K + \varepsilon)$ -equivalent to the ℓ_p basis in U . Therefore we deduce that $\mathcal{SM}_p(U, C) = \infty$.

We must consider the case in which $\mathcal{SM}_p(X_E, K) \leq \omega_1$. To obtain a contradiction, assume $\xi = \mathcal{SM}_p(U, C)$ and $\zeta = \omega^\alpha$ are such that $\zeta \xi < \mathcal{SM}_p(X_E, K)$, and in particular, ζ, ξ are countable. The argument in this case proceeds as in the previous paragraph: For any finite $S \subset E$, any $\varepsilon > 0$, and any $M \in [\mathbb{N}]$, there exists $F \in \mathcal{S}_\zeta$ and $(a_n)_{n \in F} \in S_{\ell_p^{|F|}}$ so that $\|P_S \sum_{n \in F} a_n x_{m_n}\| < \varepsilon$. If this were not so, then $(P_S x_n)_{n \in M}$ would be an ℓ_p^ξ spreading model in the finite direct sum X_S . But since $\mathcal{SM}_p(X_e) < \omega^\alpha = \zeta$, for any finite $S \subset E$, $\mathcal{SM}_p(X_S) \leq \sum_{e \in S} \mathcal{SM}_p(X_e) < \omega^\alpha$. Therefore if we choose $M \in [\mathbb{N}]$ so that $\mathcal{S}_\xi[\mathcal{S}_\zeta](M) \subset \mathcal{S}_{\zeta+\xi}$, we can choose a p -absolutely convex block (y_n) of (x_{m_n}) and disjointly supported vectors (z_n) in X_E so that $\|y_n - z_n\| < \varepsilon_n$ for all $n \in \mathbb{N}$ and so that $y_n = \sum_{i \in F_n} a_i x_{m_i}$, where $F_n \in \mathcal{S}_\zeta$. Thus the sequence (y_n) is a K - ℓ_p^ξ spreading model, and (z_n) is a C - ℓ_p^ξ spreading model for an auspicious choice of ε_n . Therefore (Πz_n) is a C - ℓ_p^ξ spreading model in U , a contradiction.

To treat the $p = 1$ case with the assumption that $\mathcal{SM}_1(X_e) \leq \omega^\alpha$ for each $e \in E$, we replace the subadditivity with the fact that admitting no $\ell_1^{\omega^\alpha}$ spreading model is a three space property. Thus any finite direct sum of spaces admitting no $\ell_1^{\omega^\alpha}$ admits no $\ell_1^{\omega^\alpha}$ spreading model.

□

We come to one final application of these methods.

Proposition 8.7. *Fix $\xi \in \mathbf{Ord}$. If for each $k \in \mathbb{N}$, X_k is a Banach space with $\mathcal{SM}_1(X_k) \leq \omega^\xi$, then $\mathcal{SM}_p((\oplus X_k)_{\ell_2}) \leq \omega^\xi$.*

Proof. Choose $\eta, \mu \in (0, 1)$ so that $\eta^2 + \mu^2 = 1$ and $2\eta^2 - 1 > \cos \theta$, where θ is the angle between (η, μ) and (μ, η) . If $(\oplus X_k)_{\ell_2}$ admits an $\ell_1^{\omega^\xi}$ spreading model, it admits one with constant $1 \leq K < \eta^{-1}$. Assume that (x_n) is a K - $\ell_1^{\omega^\xi}$ spreading model in $(\oplus X_k)_{\ell_2}$. Let $v_n = \sum_k \|\pi_{e_k}(x_n)\| e_k \in B_{\ell_2}$. By passing to a subsequence, we can assume that there exists $v \in B_{\ell_2}$ and a bounded block sequence (b_n) in ℓ_2 so that $\|v_n - (v + b_n)\| \rightarrow 0$. Then for any

free ultrafilter \mathcal{U} on \mathbb{N} and any $l \in \mathbb{N}$,

$$\begin{aligned}
l^2 K^{-2} &\leq \lim_{n_1 \in \mathcal{U}} \dots \lim_{n_l \in \mathcal{U}} \left\| \sum_{i=1}^l x_{n_i} \right\|^2 \\
&= \lim_{n_1 \in \mathcal{U}} \dots \lim_{n_l \in \mathcal{U}} \sum_k \left\| \sum_{i=1}^l \pi_{e_k}(x_{n_i}) \right\|^2 \\
&\leq \lim_{n_1 \in \mathcal{U}} \dots \lim_{n_l \in \mathcal{U}} \sum_k \left(\sum_{i=1}^l \|\pi_{e_k}(x_{n_i})\| \right)^2 \\
&= \lim_{n_1 \in \mathcal{U}} \dots \lim_{n_l \in \mathcal{U}} \left\| \sum_{i=1}^l v_{n_i} \right\|_{\ell_2}^2 \\
&= l^2 \|v\|^2 + l \lim_{n \in \mathcal{U}} \|b_n\|^2.
\end{aligned}$$

Since this holds for any $l \in \mathbb{N}$, $\eta < K^{-1} \leq \|v\|$. Choose $m \in \mathbb{N}$ so that with $v = \sum c_k e_k$, $\eta^2 < \sum_{k=1}^m c_k^2$.

Let $f : \widehat{\mathcal{S}}_{\omega^\xi} \rightarrow \{0, 1\}$ be defined by letting $f(E) = 1$ if $\min\{\|P_{[1,m]}x\| : x \in \text{co}_1(x_n : n \in E)\} \geq \mu$ and $f(E) = 0$ otherwise. If there exists $M \in [\mathbb{N}]$ so that $f|_{\widehat{\mathcal{S}}_{\omega^\xi}(M)} \equiv 1$, then $(P_{[1,m]}x_n)_{n \in M}$ is a μ^{-1} - $\ell_1^{\omega^\xi}$ spreading model in the finite direct sum $(\oplus_{i=1}^m X_k)_{\ell_2^m}$, a contradiction. Therefore by Lemma 3.5 there exists $F_1 < F_2 < \dots$ so that $f(F_i) = 0$ for all $i \in \mathbb{N}$ and so that for each $F \in \mathcal{S}_{\omega^\xi}$, $\cup_{i \in F} F_i \in \mathcal{S}_{\omega^\xi}$. Choose $x \in \text{co}_1(x_n : n \in F_2)$ so that $\|P_{[1,m]}x\| < \mu$ and for each $i > 2$, choose $n_i \in F_i$. Then since $(2, i) \in \mathcal{S}_{\omega^\xi}$, $F_2 \hat{\sim} n_i \subset F_2 \hat{\sim} F_i \in \mathcal{S}_{\omega^\xi}$ for all $i > 2$. This means that for all $i > 2$,

$$\begin{aligned}
4\eta^2 &\leq \|x + x_{n_i}\|^2 = \|P_{[1,m]}(x + x_{n_i})\|^2 + \|P_{(m,\infty)}(x + x_{n_i})\|^2 \\
&\leq \|P_{[1,m]}x\|^2 + 2\|P_{[1,m]}x\|\|P_{[1,m]}x_{n_i}\| + \|P_{[1,m]}x_{n_i}\|^2 \\
&\quad + \|P_{(m,\infty)}x\|^2 + 2\|P_{(m,\infty)}x\|\|P_{(m,\infty)}x_{n_i}\| + \|P_{(m,\infty)}x_{n_i}\|^2 \\
&\leq 2 + 2u \cdot v.
\end{aligned}$$

where

$$u = (\|P_{[1,m]}x\|, (1 - \|P_{[1,m]}x\|)^2), \quad v = (\|P_{[1,m]}x_{n_i}\|, (1 - \|P_{[1,m]}x_{n_i}\|)^2)^{1/2} \in \mathbb{R}^2.$$

But since $\|P_{[1,m]}x\| < \mu$ and for sufficiently large i , $\|P_{[1,m]}x_{n_i}\| \approx (\sum_{k=1}^m c_k^2)^{1/2} > \eta$, the angle ϕ between u and v is greater than the angle θ between (μ, η) and (η, μ) . Then we deduce

$$2\eta^2 - 1 \leq u \cdot v = \cos \phi < \cos \theta < 2\eta^2 - 1,$$

and this contradiction finishes the proof. \square

With this, we deduce that for any $0 < \xi < \omega_1$ and $\xi_n \uparrow \omega^\xi$, the direct sum $(\oplus X_{\xi_n})_{\ell_2}$ of the Schreier spaces X_{ξ_n} admits 1 - ℓ_1^ζ spreading models for all $\zeta < \omega^\xi$, but does not admit an $\ell_1^{\omega^\xi}$ spreading model. As usual, we can deduce that the dual admits 1 - c_0^ζ spreading

models for all $\zeta < \omega^\xi$ with no $c_0^{\omega^\xi}$ spreading model, and the p -convexification admits a 1 - ℓ_p^ζ spreading model for each $\zeta < \omega^\xi$ but no $\ell_p^{\omega^\xi}$ spreading model. Thus we see that the supremum $\mathcal{SM}_p(X) = \sup_{K \geq 1} \mathcal{SM}_p(X, K)$ may be attained, and in fact the examples here give for each $1 \leq p \leq \infty$ an example of some X so that $\omega^\xi = \mathcal{SM}_p(X) = \mathcal{SM}_p(X, 1)$.

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